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Dynamics and E-Learning

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Abstract

The aim of this paper is to demonstrate how the OViSS framework can be used to create learning components for economic teaching. In particular, we are interested in applications which go beyond merely transferring textbooks into 'html'-files.

Especially, we want to show that by using CAL-programs actions and iterations can be demonstrated which could hardly be shown without this technology. The components we will present were developed for different topics of dynamic theory:

- Population Dynamics
- Dynamic Games
- Market Dynamics

In the lecture I will especially demonstrate, how the orbits of a system of two nonlinear first order differential equations can be constructed with the help of a spreadsheet and I will show, that many interesting problems in economics can thus be investigated.

1 Introduction

1.1 Intention

Many discussions on the topic of the 'electronic classroom' have resulted in heavy criticism. Accordingly, critics would prefer completely new computer based courses to be designed instead of the customary practice of merely adding a disc to a standard textbook. We have designed the OViSS framework with this idea in mind. At the time of writing, the first public version of OViSS has not yet been available. Thus, in section 0 we present a preview version of OViSS, which is not full-featured. Nevertheless, we are confident that its essential features are clearly discernible. In order to demonstrate the program, different courses in economics have been chosen to serve as an example. However, the employed concepts and ideas can easily be applied to other areas of economics as well as other fields of science. The Open Virtual Study System (OViSS) [Reiß und Menkhoff 2002] is a key component of the VORMS project. The technological aspects of this project are presented in [Menkhoff und Reiß 2002]

1.2 Interactive computer-aided Learning

Students working with an interactive computer-based training program should be able to perform the following tasks:

1. Read statements on a special problem, either printed out or preferably displayed on the screen.
2. Solve problems and decide on solutions, which are subsequently evaluated by the program.
3. If problems with tasks arise, a context-sensitive help function should be available.
4. Enter data into charts, draw curves on diagrams, locate the correct positions of curves on the graphs, and finally interpret the diagrams dealt with.
5. Observe changes in diagrams, resulting, e.g., from different time periods, alteration in price, information progress, or other variables.

1.3 The Structure of OViSS

The Principles

In principle, OViSS consists of three parts. 1. An actually empty frame that provides a menu system, a toolbar and several different functionalities for managing the contents. 2. An extensible pool of program files, which provide the means to interactively present nearly any kind of learning objects. 3. A collection of learning objects coded in QTI, a commonly accepted standard (see [Menkhoff und Reiß 2002]). Due to this open architecture, it is comparatively easy to integrate a wide range of applications into the system. The program can handle components of applications that it did not know at the time of compilation and that were possibly created even at a later date. OViSS exclusively uses XML files for data management. Due to the deployment of this standard, an international data exchange is easily feasible.

The structure in detail

After starting the program a competence centre is generated that contains a number of objects that serve as "experts". These experts manage the users' access to the system, the display of tables of contents and the loading and unloading of contents. When a user has got access to the system after logging in as a known user, the frame of the system is generated and the components with which the user had terminated his last session are loaded; if the user is a new user, he will be presented the starting page of a demonstration. The individual learning components are grouped in books and chapters (figure 1). A book consists of different chapters that comprise several sections. The user can choose the books he wants to work with out of a list of books. The system is able to identify the books and their contents by their locations. A location can be a path indication on the local computer or a URI in the Internet. Thus, the user is not restricted in his work to the components installed on his own computer.

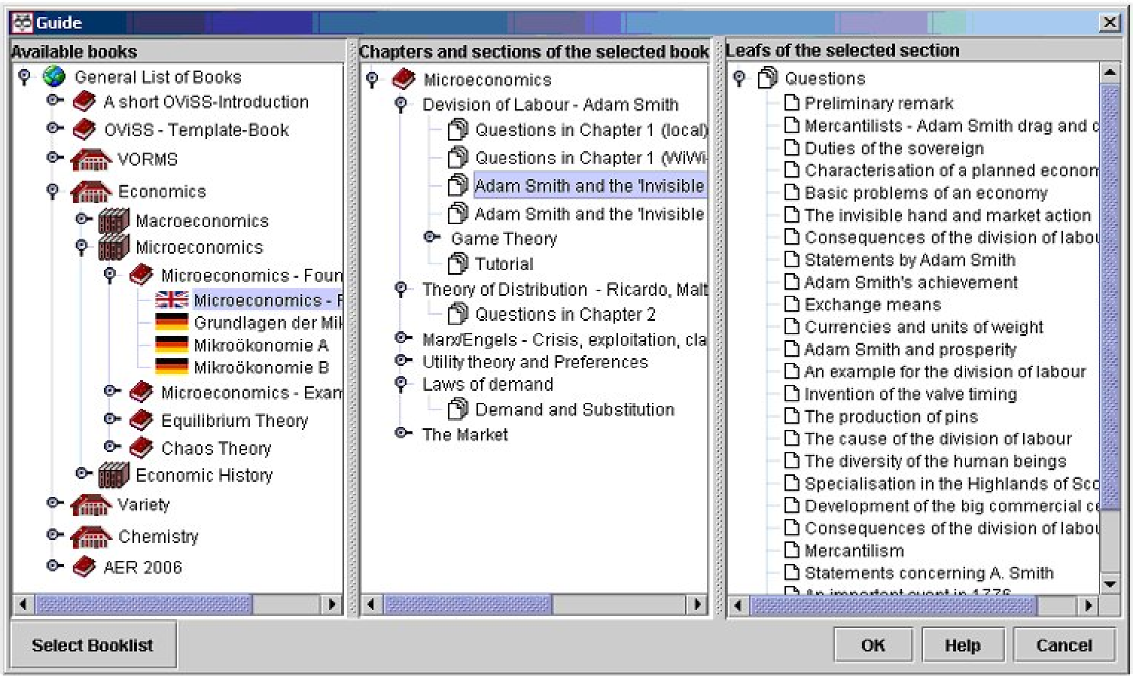


Figure 1: The Guide

2 Differential equations

Important examples within dynamic theory are based on linear and nonlinear differential equations (see, e.g., [Glendinning 1994]). We start with some examples from linear differential equations.

2.1 Linear differential equations

In the OViSS learning system we have implemented many lectures on this topic where the student can acquire knowledge about these equations. In some lectures the user can watch the developing trajectories from certain starting points (entered by mouse click). Doing so, he/she can study the general trajectories in a phase plane.

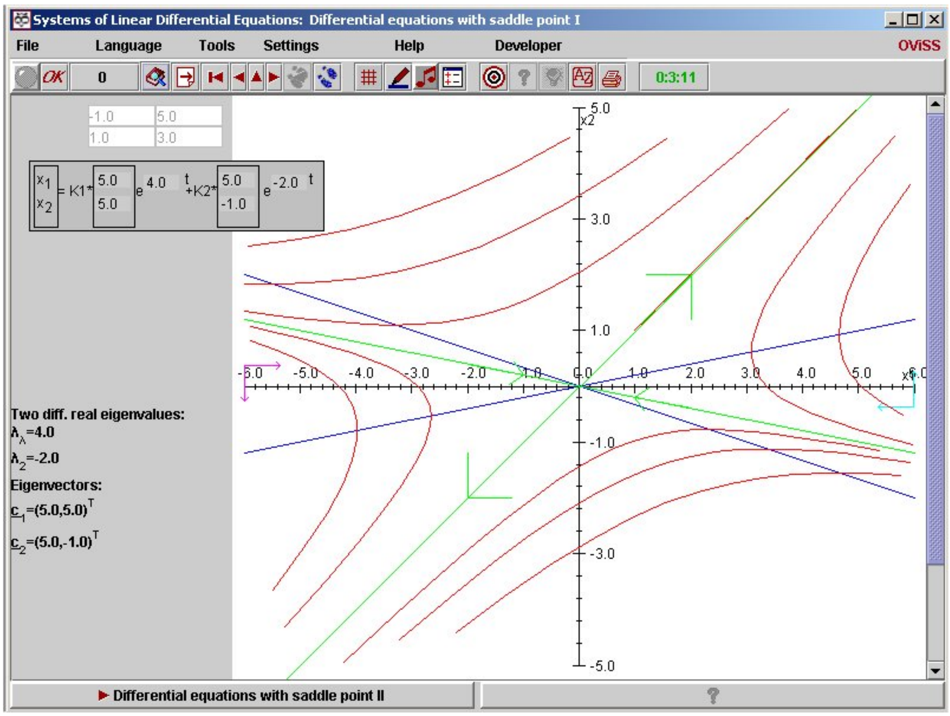


Figure 2: Linear Differential Equations with Saddle Point

The examples - shown in figures 2 and 3 - can easily be altered in such a way that the user starts with an empty (or nearly empty) phase portrait and is asked to find the phase portrait himself by clicking on the screen and watching the evolving trajectories. This feature clearly shows one of the many advantages of computer-based teaching.

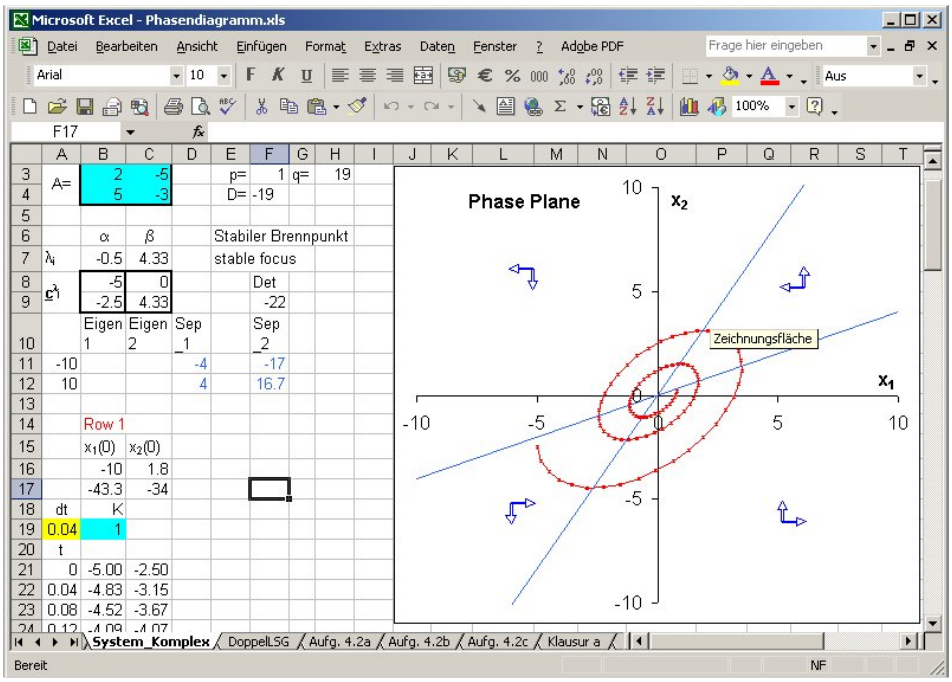


Figure 3: Linear Differential Equations with Stable Focus

2.2 Nonlinear differential equations

Solutions to nonlinear differential equations are much harder to obtain. Often it is impossible to find the exact solution.
Nevertheless there are many examples in economic theory, where the models show nonlinear behaviour.
Examples are found in

- market dynamics
- evolutoric games
- population dynamics
- chaos theory

In Figure 4 one finds a well-known example of chaos-theory.

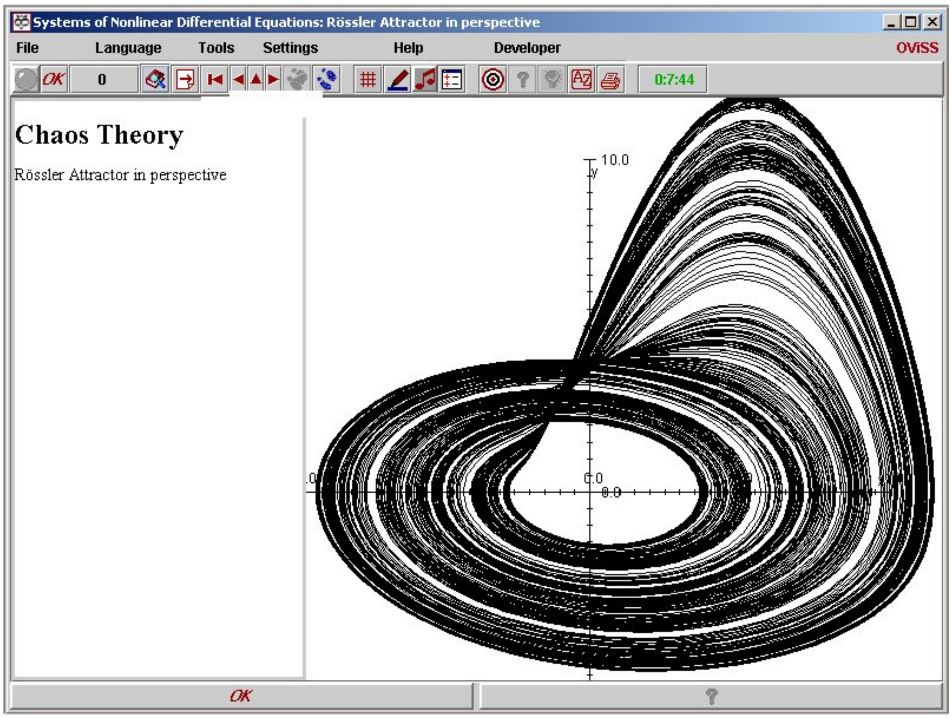


Figure 4: Rössler Attractor

We now turn our attention to a special field of economic theory, where the students have to deal with nonlinear differential equations:
The Lotka-Volterra population-dynamics.

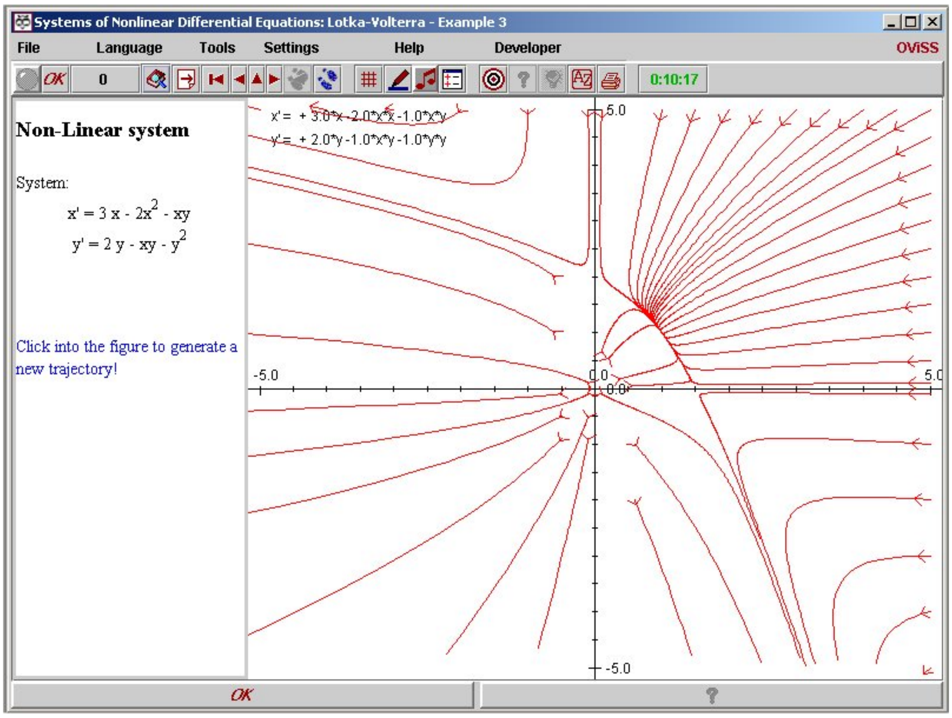


Figure 5: Lotka-Volterra as an applet

2.3 Lotka Volterra Population Dynamics

We look at two species x_1 and x_2, who live in a habitat like e.g. a pool, a lake or an island. We assume that they grow exponentially for small populations, yet are limited by the size of the habitat for large ones. In addition the size of one population has a (positive or negative) effect on the other one.

We therefore examine the following model

$$\begin{aligned}\dot{x}_1 &= x_1(b_1 + a_{11}x_1 + a_{12}x_2)\\ \dot{x}_2 &= x_2(b_2 + a_{21}x_1 + a_{22}x_2)\end{aligned}$$

This is a system of two nonlinear first order differential equations.

For $x_2 = 0$ the first equation becomes

$$\dot{x}_1 = x_1(b_1 + a_{11}x_1) = b_1x_1 + a_{11}x_1^2$$

For small x_1 we can neglect x_1^2 and the relation becomes:

$$\dot{x}_1 = b_1x_1$$

For $b_1 > 0$ one gets a positive exponential growth - the species nourishes e.g. by gras. If however the species is a predator, who lives on the second, the prey, a $x_2 = 0$ will cause starvation and hence negative exponential growth and therefore $b_1 < 0$.

If x_1 is not small we can not neglect it and the term x_1^2 has to be considered. As the living space is finite, overpopulation will reduce growth and we get $a_{11} < 0$.

If $x \neq 0$ and $y \neq 0$ the coefficients $a_{ij}(i \neq j)$ measure the effects of one population to the other:

Is $a_{12} < 0$ and $a_{21} < 0$, growth of one population is smaller the larger the other one is. Hence the populations compete for a scarce resource as e.g. sheep and goat do on a grassy island.

If $a_{12} > 0$ as well as $a_{21} > 0$, growth of one population is higher, the larger the other population. We have a symbiotic connection as e.g. with bees and apples.

If a_{12} and a_{21} differ in sign, we have got a predator-prey connection as with foxes and hares. Is e.g, $a_{12} > 0$ and $a_{21} < 0$ then x_1 (foxes) growth faster the higher the population of the second (the hares), while a high number of foxes reduce the growth rate of hares hence $a_{21} < 0$.

2.4 Euler's method

We examine two differential equations

$$\dot{x} = \frac{\partial x}{\partial t} = f(t, x, y)$$

and

$$\dot{y} = \frac{\partial y}{\partial t} = g(t, x, y)$$

Starting from an initial value (x_0, x_0) we can plot the solutions depending on the time $t = 1, 2, \ldots$ by calculating x(t), y(t).
The path, that we get, is called a trajectory or an orbit of the system.

To get approximations we rearrange

$$\frac{\Delta x}{\Delta t} = f(t,x,y)$$
$$\frac{\Delta y}{\Delta t} = g(t,x,y)$$

hence

$$\Delta x = \Delta t \cdot f(t,x,y)$$
$$\Delta y = \Delta t \cdot g(t,x,y)$$

Starting from an initial value (x_0, x_0) we can plot the solutions depending on the time $t = 1, 2, \ldots$ by calculating

$$x_{t+1} = x_t + \Delta x = x_t + \Delta t \cdot f(t,x,y)$$
$$y_{t+1} = y_t + \Delta y = y_t + \Delta t \cdot g(t,x,y)$$

This method is the so called 'Euler's method'. It is very easy to implement, the students can do that themselves and by that study the behaviour of nonlinear differential equations. Therefore we use this method with the help of spreadsheets like 'Excel'. As we will see, this method can have it's problems.

2.5 Solving with Excel

Let us assume that the differential equations are of the form

$$\dot{x} = a + a_x \cdot x + a_y \cdot y + a_{xx} \cdot x \cdot x + a_{xy} \cdot x \cdot y + a_{yy} \cdot y \cdot y + a_{xxx} \cdot x \cdot x \cdot x + a_{xxy} \cdot x \cdot x \cdot y + a_{xyy} \cdot x \cdot y \cdot y + a_{yyy} \cdot y \cdot y \cdot y$$

$$\dot{y} = b + b_x \cdot x + b_y \cdot y + b_{xx} \cdot x \cdot x + b_{xy} \cdot x \cdot y + b_{yy} \cdot y \cdot y + b_{xxx} \cdot x \cdot x \cdot x + b_{xxy} \cdot x \cdot x \cdot y + b_{xyy} \cdot x \cdot y \cdot y + b_{yyy} \cdot y \cdot y \cdot y$$

hence first order cubic differential equations. This type is rather common in economics, it comprises those of the Lotka-Volterra population models as well as many examples in chaos theory and also those of evolutionary games. If we construct a general Excel-solution, we can use it again and again. This, however, is easily done.
We write a sheet with the following contents and name the cells A8:J8 with the entries for the variables a, a_x etc. in the line above those cells. The same has to be done with the variables b, b_x etc. in cells A11:J11.

	A	B	C	D	E	F	G	H	I	J
7	a	a_x	a_y	a_xx	a_xy	a_yy	a_xxx	a_xxy	a_xyy	a_yyy
8	0	3	-5	0	0	0	0	0	0	0
9										
10	b	b_x	b_y	b_xx	b_xy	b_yy	b_xxx	b_xxy	b_xyy	b_yyy
11	0	5	-3	0	0	0	0	0	0	-1
12										
13	Step d=	0.1								
14										
15	t	x	y							
16	0	0.4	0.2							
17		((*))	((**))							

In cell B17 we feed the following formula:
=B16+d*(a+a_x*B16+a_y*C16+a_xx*B16ˆ2+a_xy*B16*C16+a_yy*C16ˆ2+a_xxx*B16ˆ3+a_xyy*B16*C16ˆ2+a_xxy*B16ˆ2*C16+a_yyy*C16ˆ3)

and in C17:
=C16+d*(b+b_x*B16+b_y*C16+b_xx*B16ˆ2+b_xy*B16*C16+b_yy*C16ˆ2+b_xxx*B16ˆ3+b_xxy*B16ˆ2*C16+b_xyy*B16*C16ˆ2+b_yyy*C16ˆ3)

We copy the cells B17:C17 down to e.g. B217:C217 and get the first 200 points of the orbit of the system, calculated by Euler's method. These values can be fed into a diagram and one gets the following sheet:

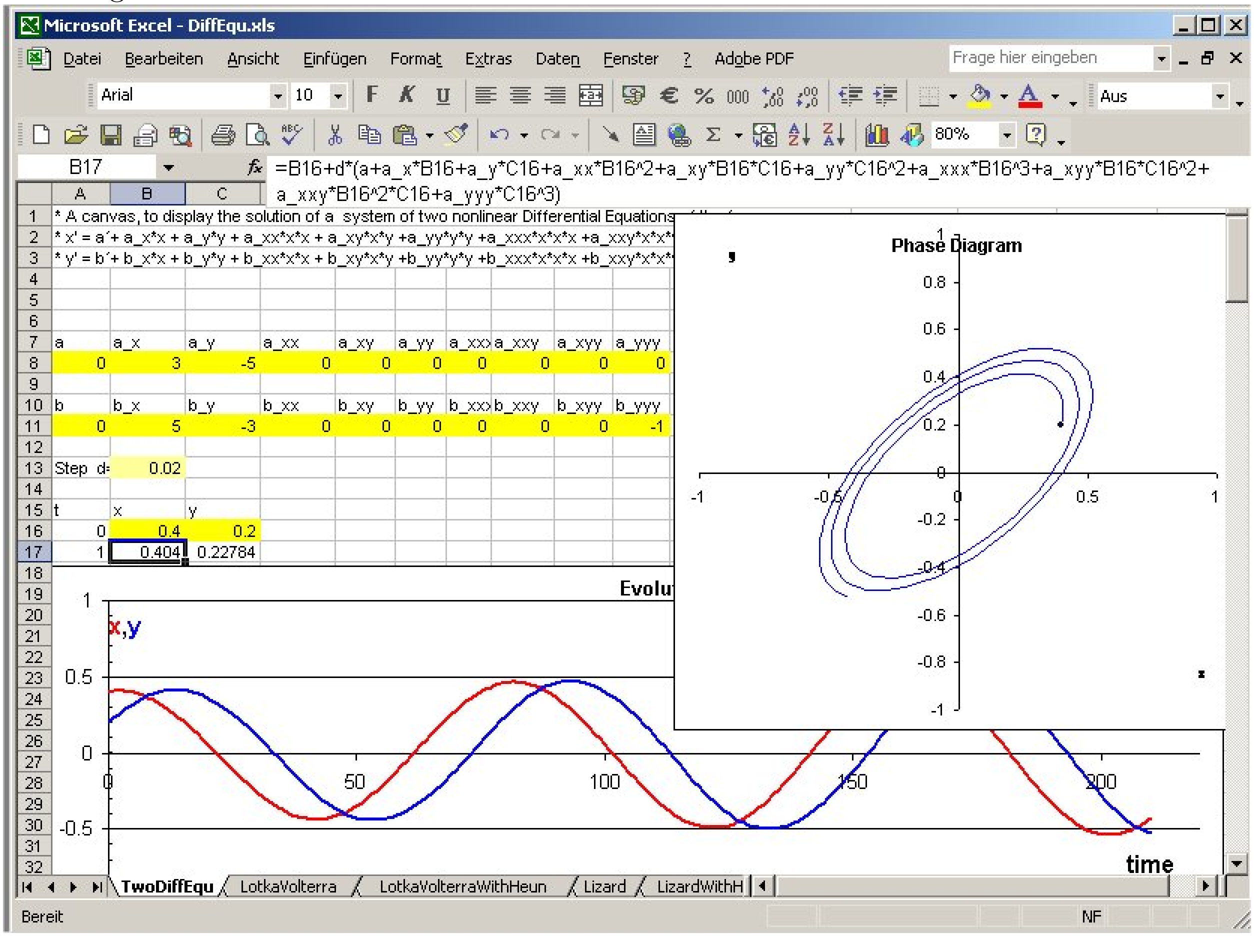


Figure 6: The Euler method in Excel

We now can use this method to study dynamic developments in economics. First we will analyse Lotka-Volterra models.

2.6 Stationary points

We want to find stationary points. There we have:

$$\begin{aligned} \dot{x}_1 &= 0 \\ \dot{x}_2 &= 0 \end{aligned}$$

and hence

$$\begin{aligned} 0 &= x_1(b_1 + a_{11}x_1 + a_{12}x_2) \\ 0 &= x_2(b_2 + a_{21}x_1 + a_{22}x_2) \end{aligned}$$

There are four solutions

1. Solution

$$x_1 = 0 \quad x_2 = 0$$

This is a trivial solution, with both populations constantly zero.

2. Solution

$$\begin{array}{rcl} x_1 &=& 0 \qquad x_2 \neq 0 \\ 0 &=& x_2(b_2 + a_{22}x_2) \\ 0 &=& b_2 + a_{22}x_2 \\ x_2 &=& -\frac{b_2}{a_{22}} \qquad (\text{if } a_{22} \neq o) \end{array}$$

3. Solution

$$\begin{array}{rcl} x_2 &=& 0 \qquad x_1 \neq 0 \\ 0 &=& x_1(b_1 + a_{11}x_1) \\ 0 &=& b_1 + a_{11}x_1 \\ x_1 &=& -\frac{b_1}{a_{11}} \qquad (\text{if } a_{11} \neq 0) \end{array}$$

4. Solution

$$\begin{array}{l} x_1 \neq 0 \qquad x_2 \neq 0 \\ 0 = b_1 + a_{11}x_1 + a_{12}x_2 \\ 0 = b_2 + a_{21}x_1 + a_{22}x_2 \\ A\underline{\boldsymbol{x}} = -\underline{\boldsymbol{b}} \end{array}$$

With Cramer's rule one gets

$$x_1 = -\frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}} = -\frac{b_1a_{22} - b_2a_{12}}{a_{11}a_{22} - a_{12}a_{21}}$$

$$x_2 = -\frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}} = -\frac{b_2a_{11} - b_1a_{21}}{a_{12}a_{22} - a_{12}a_{21}}$$

These solution only exist, if $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \neq 0$.

2.7 Examples

Example 1

$$\dot{x}_1 = x_1(3 - x_1 - 2x_2)$$
$$\dot{x}_2 = x_2(2 - x_1 - x_2)$$

1. solution
$x_1 = 0,\ x_2 = 0$

2. solution
$x_1 = 0,\ x_2 = -\frac{b_2}{a_{22}} = -\frac{2}{-1} = 2$

3. solution
$x_1 = -\frac{b_1}{a_{11}} = -\frac{3}{1} = 3,\ x_2 = 0$

4. solution
$x_1 = -\frac{3(-1)-2(-2)}{(-1)(-1)-(-2)(-1)} = -\frac{1}{-1} = 1,$
$x_2 = -\frac{(-1)\cdot 2-(-1)\cdot 3}{-1} = -\frac{1}{-1} = 1$

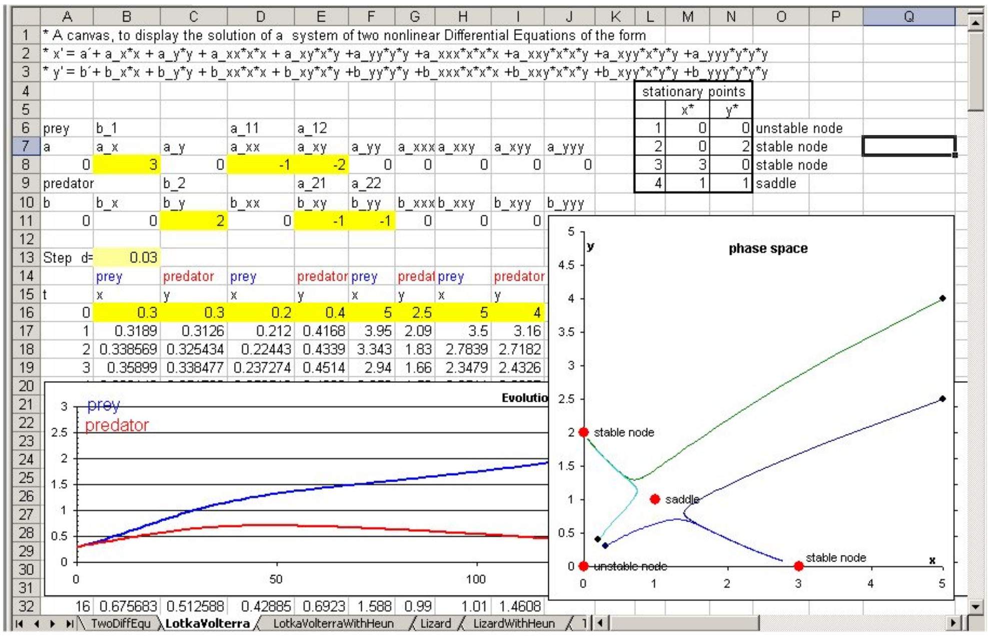


Figure 7: Lotka Volterra – Example 1 with Excel

For the examination of the stability of the solutions we have to form the Jacobian matrix of the two differential equations

$$\dot{x}_1 = f_1(x_1, x_2)$$
$$\dot{x}_2 = f_2(x_1, x_2)$$

This matrix is

$$J = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{pmatrix} = \begin{pmatrix} b_1 + 2a_{11}x_1 + a_{12}x_2 & a_{12}x_1 \\ a_{21}x_2 & b_2 + 2a_{21}x_1 + 2a_{22}x_2 \end{pmatrix}$$

Depending on the eigenvalues of that Jacobian we have

- a focus, if the eigenvalues are complex
- a saddle, if the real eigenvalues differ in sign
- a stable node, if both real eigenvalues are less than zero
- an instable node, if both real eigenvalues are greater than zero.

This calculation is done in the spreadsheet and the results are shown on the spreadsheet. In the diagram inside the spreadsheet the stationary points are denoted by red dot together with a label giving the type of the point.
In this example we have an unstable node at (0, 0), a saddle at (1, 1), and two stable nodes at (3, 0) and (0, 2): Depending on (positive) initial conditions one and only one of the populations will die out.

Example 2

$$\begin{aligned}\dot{x}_1 &= x_1(3 - x_1 - x_2)\\ \dot{x}_2 &= x_2(2 - x_1 - x_2)\end{aligned}$$

1. Solution
$x_1 = 0,\ x_2 = 0$

2. Solution
$x_1 = 0,\ x_2 = -\frac{2}{-1} = 2$

3. Solution
$x_2 = 0,\ x_1 = -\frac{3}{-1} = 3$

4. Solution does not exist, as $\begin{vmatrix} a_{21} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = 0$.

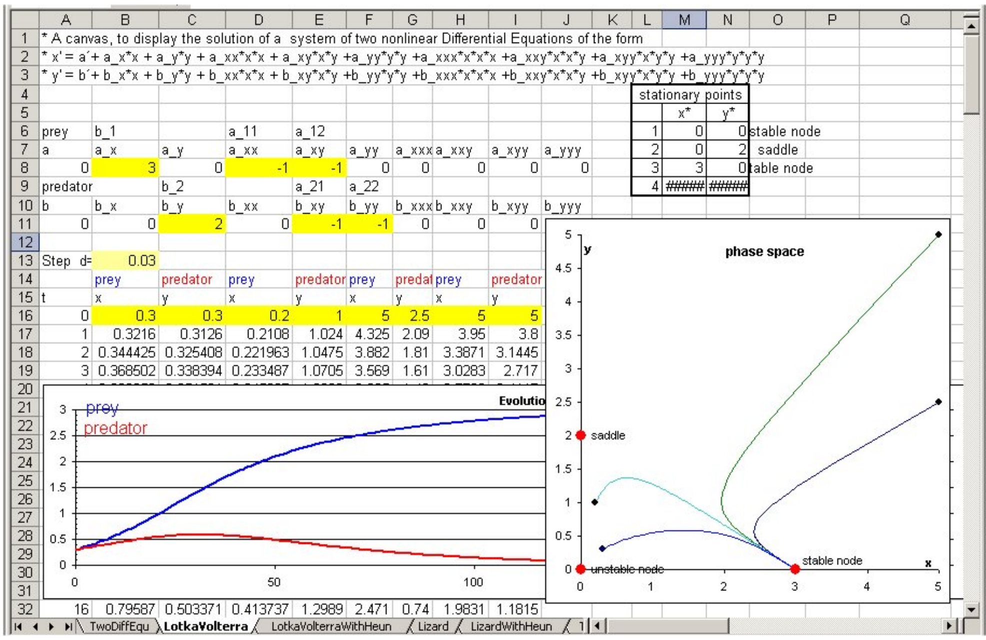


Figure 8: Lotka Volterra – Example 2 with Excel

Example 3

$$\begin{aligned}\dot{x}_1 &= x_1(3 - 2x_1 - x_2)\\ \dot{x}_2 &= x_2(2 - x_1 - x_2)\end{aligned}$$

1. Solution
$x_1 = 0,\ x_2 = 0$

2. Solution
$x_1 = 0,\ x_2 = -\frac{2}{-1} = 2$

3. Solution
$x_2 = 0,\ x_1 = -\frac{3}{-1} = 3$

4. Solution
$x_1 = -\frac{3(-1)-2(-1)}{(-2)(-1)-(-1)(-1)} = -\frac{-1}{1} = 1$
$x_2 = -\frac{(-2)\cdot 2-(-1)\cdot 3}{1} = -\frac{-1}{1} = 1$

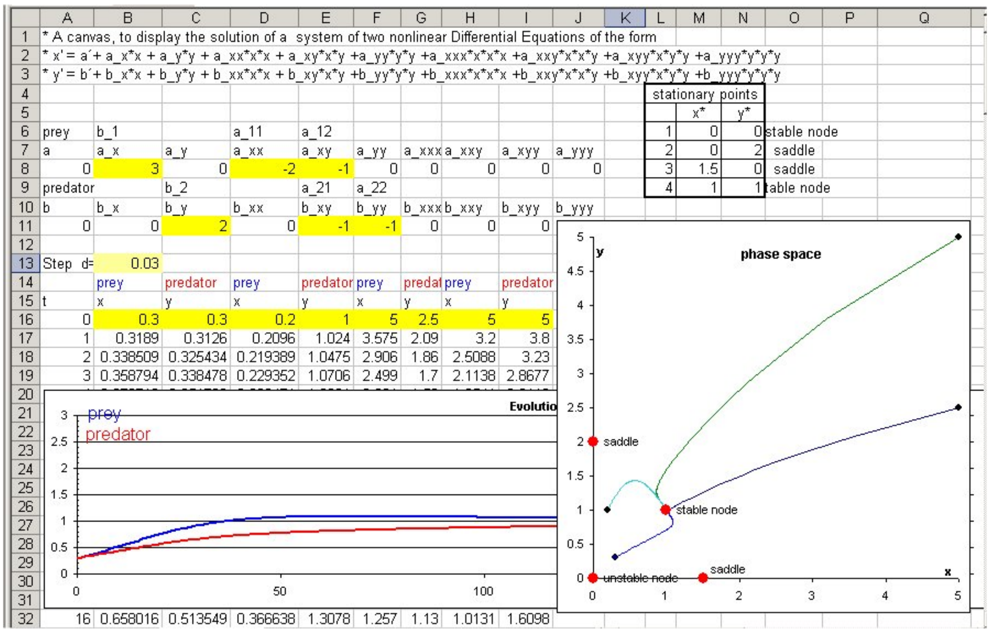


Figure 9: Lotka Volterra – Example 3 with Excel

Example 4

$$\begin{aligned}\dot{x}_1 &= x_1(3 - 2x_1 - 2x_2)\\ \dot{x}_2 &= x_2(2 - x_1 - x_2)\end{aligned}$$

1. Solution
$x_1 = 0 \qquad x_2 = 0$

2. Solution
$x_1 = 0 \qquad x_2 = -\frac{2}{-1} = 2$

3. Solution
$x_2 = 0 \qquad x_2 = -\frac{3}{-2} = 1,5$

4. Solution does not exist.

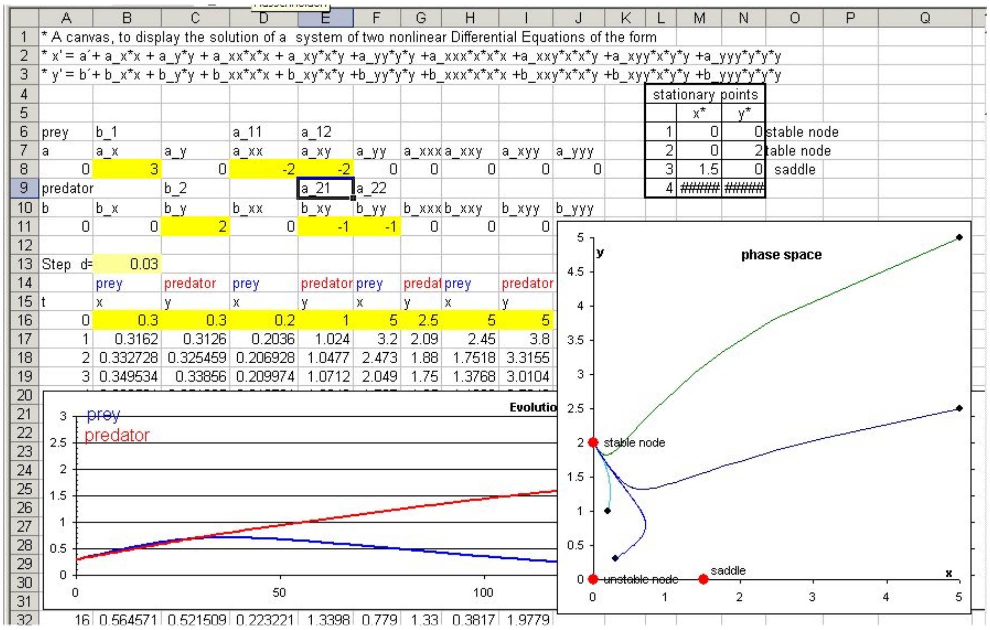


Figure 10: Lotka Volterra – Example 4 with Excel

Example 5

$$\begin{aligned}\dot{x}_1 &= x_1(1 - 3x_2)\\ \dot{x}_2 &= x_2(-1 + 3x_1)\end{aligned}$$

1. Solution
$x_1 = 0 \qquad x_2 = 0$

2. Solution
does not exist.

3. Solution
does not exist.

4. Solution

$x_1 = -\frac{3 \cdot 0 - (-3)(-3)}{0 \cdot 0 - (-3) \cdot 3} = -\frac{-9}{9} = 1$

$x_2 = -\frac{0 \cdot (-3) - 3 \cdot 3}{9} = -\frac{-9}{9} = 1$

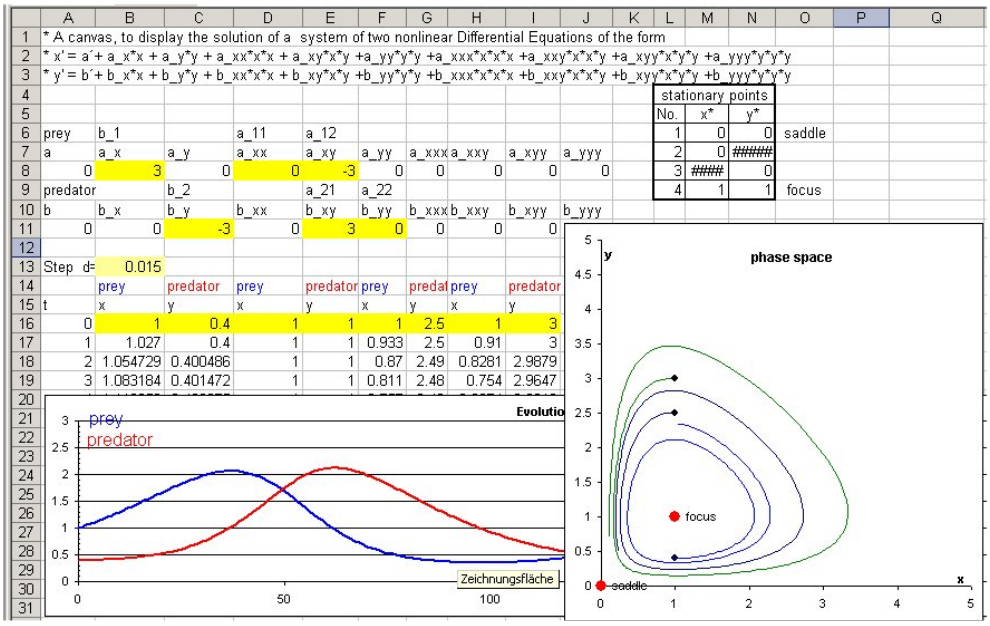


Figure 11: Lotka Volterra – Example 5 with Excel

We examine the stability at point (1, 1)

$$J = \begin{pmatrix} b_1 + 2a_{11}x_1 + a_{22}x_2 & a_{12}x_1 \\ a_{21}x_2 & b_2 + a_{21}x_1 + a_{22}x_2 \end{pmatrix} = \begin{pmatrix} 1 + 2 \cdot 0 \cdot \frac{1}{3} - 3 \cdot \frac{1}{3} & -3\frac{1}{3} \\ 3 \cdot \frac{1}{3} & -1 + 3 \cdot \frac{1}{3} + 0 \cdot \frac{1}{3} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Eigenvalues are found by determining the roots of the determinant $|J - \lambda I|$:

$$|J - \lambda I| = \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = 0$$

$$\lambda^2 + 1 \quad = 0$$

$$\lambda \quad = \pm\sqrt{-1}$$

The eigenvalues are complex, the orbits oscillate around a stationary point, we have got a focus.

3 Evolutoric game theory

3.1 Asymmetric trade

In [Sieg 2005, p. 73] there is the following example:

There are sellers and buyers. A part of the buyers as well as a part of the buyers is aggressive and the other non-aggressive. If a buyer meets a seller, there possibly will be a trade depending on the type of two traders. An aggressive buyer and a an aggressive seller will not come to terms while an aggressive buyer will exploit the non-aggressive seller and vice versa. Two non-aggressive traders will share the profit. With that, we will get the game matrix on the right.

		buyer	
		aggressiv	non-aggress.
seller	aggress.	0 (seller), 0 (buyer) (I)	3 (seller), 1 (buyer) (II)
	non-aggr.	1 (seller), 3 (buyer) (III)	2 (seller), 2 (buyer) (IV)

Figure 12: Asymmetric trade

Altogether there are x aggressive sellers and y aggressive buyers. Then we get the utility of a seller from the utility of his two strategies:

$$\begin{aligned} u(\text{aggr. seller}) &= y \cdot 0 + (1-y) \cdot 3 = 3 - 3y \\ u(\text{non aggr. seller}) &= y \cdot 1 + (1-y) \cdot 2 = 2 - y \end{aligned}$$

Accordingly we get for a buyer:

$$\begin{aligned} u(\text{aggr. buyer}) &= x \cdot 0 + (1-x) \cdot 3 = 3 - 3x \\ u(\text{non aggr. buyer}) &= x \cdot 1 + (1-x) \cdot 2 = 2 - x \end{aligned}$$

We get as the average payment for the sellers, x of whom choose the strategy "agressive":

$$\begin{aligned} & x \cdot u(\text{aggr. seller}) + (1-x)u(\text{non aggr. seller}) \\ &= x\{0 \cdot y + 3 \cdot (1-y)\} + (1-x)\{1 \cdot y + 2 \cdot (1-y)\} \\ &= \begin{bmatrix} x & 1-x \end{bmatrix} \begin{bmatrix} 0 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} y \\ 1-y \end{bmatrix} \end{aligned}$$

We get as the average payment for the buyers, y of whom choose the strategy "agressive":

$$\begin{aligned} & y \cdot u(\text{aggr. Buyer}) + (1-y)u(\text{non aggr. Buyer}) \\ &= y \cdot \{0 \cdot x + 3(1-x)\} + (1-y)\{1 \cdot x + 2(1-x)\} \\ &= \begin{bmatrix} y & 1-y \end{bmatrix} \begin{bmatrix} 0 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ 1-x \end{bmatrix} \end{aligned}$$

We get replicator equations by calculating the changes of the values of x resp. y.

Those changes can be found by comparing the payments for the player playing strategy 'agressive' with the average payment.

$$\begin{aligned} \dot{x} &= x\Big[0 \cdot y + 3(1-y) - x\{0 \cdot y + 3(1-y)\} - (1-x)\{1y + 2(1-y)\}\Big] \\ &= x(1-x)(1-2y) \end{aligned}$$

In the same way we find:

$$\dot{y} = y(1-y)(1-2x)$$

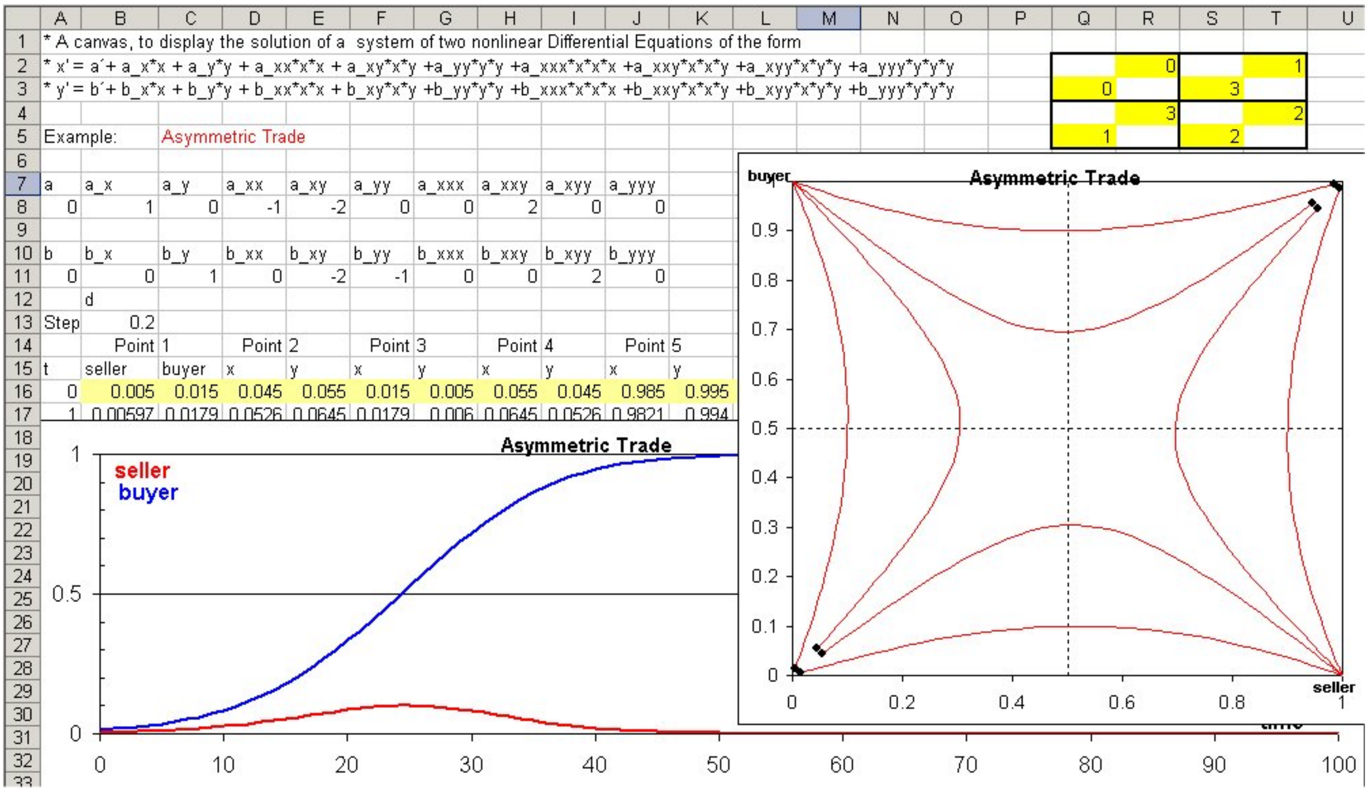


Figure 13: Asymmetric Trade

3.2 Generalizing

We want to generalize this example so that we can construct different orbits for different types of games.

Let X be the first player with the alternatives X_1 and X_2. Equivalently the second player Y has alternatives Y_1 and Y_2.

There are altogether x players of type X and y players of type Y. Then we get the utility of a player of type X from the utility of his two strategies:

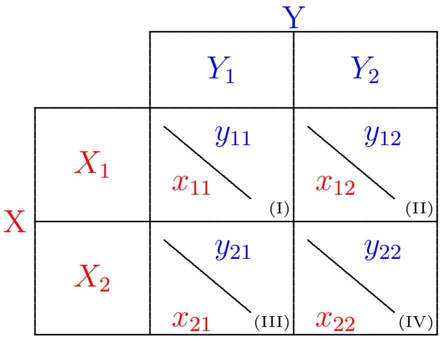


Figure 14: Asymmetric Trade

$$u(X_1) = y \cdot x_{11} + (1-y)x_{22}$$

$$u(X_2) = y \cdot x_{21} + (1-y)x_{22}$$

Using matrix we can write:

$$\underline{\boldsymbol{u}}\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \stackrel{\text{def}}{=} \begin{pmatrix} u(X_1) \\ u(X_2) \end{pmatrix} = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} \begin{bmatrix} y \\ 1-y \end{bmatrix}$$

Equivalently one gets for the utility of player Y (note however, that now columns have to be used)

$$u(Y_1) \quad = \quad xy_{11} \quad + \quad (1-x)y_{21}$$

$$u(Y_2) \quad = \quad xy_{12} \quad + \quad (1-x)y_{22}$$

Using matrix we can write:

$$\underline{\boldsymbol{u}}\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} \stackrel{def}{=} \begin{pmatrix} u(Y_1) \\ u(Y_2) \end{pmatrix} = \begin{bmatrix} y_{11} & y_{21} \\ y_{12} & y_{22} \end{bmatrix} \begin{bmatrix} x \\ 1-x \end{bmatrix}$$

The average payoff for players of type X, with x playing strategy X_1, result in

$$x \cdot u(X_1) + (1-x)u(X_2)$$

$$= \quad x\left\{yx_{11} + (1-y)x_{12}\right\} + (1-x)\left\{yx_{21} + (1-y)x_{22}\right\}$$

$$= [x \;\; 1-x] \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} \begin{bmatrix} y \\ 1-y \end{bmatrix}$$

The average payoff for players of type Y, with y playing strategy Y_1, result in

$$y \cdot u(Y_1) + (1-y)u(Y_2)$$

$$= \quad y \cdot \left\{xy_{11} + (1-x)y_{21}\right\} + (1-y)\left\{x \cdot y_{12} + (1-x)y_{22}\right\}$$

$$= [y \;\; 1-y] \begin{bmatrix} y_{11} & y_{21} \\ y_{12} & y_{22} \end{bmatrix} \begin{bmatrix} x \\ 1-x \end{bmatrix}$$

Replicator-equations result, by determining, how the quantities x resp. y change.

These changes arise by comparing the payoff to player X with strategy X_1 (resp. player Y with Y_1) with the average payoff:

$$\dot{x} \quad = x\Big[y \cdot x_{11} + (1-y)x_{12} - x\left\{yx_{11} + (1-y)x_{12}\right\} - (1-x)\left\{yx_{21} + (1-y)x_{22}\right\}\Big]$$

Using matrices, one gets:

$$\dot{x} \quad = x\left\{[1 \;\; 0]\begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} \begin{bmatrix} y \\ 1-y \end{bmatrix} - [x \;\; 1-x]\begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} \begin{bmatrix} y \\ 1-y \end{bmatrix}\right\}$$

$$= x\Big[x_{12} + (x_{11} - x_{12}) - x\left\{x_{12} + (x_{11} - x_{12})y\right\} - \left\{x_{22} + (x_{21} - x_{22})y\right\} + x\left\{y_{22} + (x_{21} - x_{22})y\right\}\Big]$$

$$\dot{x} \quad = x\Big[(x_{12} - x_{22}) + (x_{11} - x_{12} - x_{21} + x_{22})y - (x_{12} - x_{22})x - (x_{11} - x_{12} - x_{21} + x_{22})xy\Big]$$

$$= (x_{12} - x_{22})x + (x_{11} - x_{12} - x_{21} + x_{22})xy - (x_{12} - x_{22})xx - (x_{11} - x_{12} - x_{21} + x_{22})xxy$$

In the example of the "asymmetric trade" one gets:

$$\dot{x} \quad = 1x - 2xy - 1xx + 2xxy$$

that coincides with the calculated result.

For y one gets as replicator equation:

$$\dot{y} = y\Big[xy_{11} + (1-x)y_{21} - y\{xy_{11} + (1-x)y_{21}\} - (1-y)\{xy_{12} + (1-x)y_{22}\}\Big]$$

$$= y\Big[y_{21} + (y_{11} - y_{21})x - y\{y_{21} + (y_{11} - y_{21})x\} - \{y_{22} + (y_{12} - y_{22})x\} + y\{y_{22} + (y_{12} - y_{22})x\}\Big]$$

$$= (y_{21} - y_{22})y + (y_{11} - y_{21} - y_{12} + y_{22})xy - (y_{21} - y_{22})yy - (y_{11} - y_{21} - y_{12} + y_{22})yyx$$

In the example of the "asymmetric trade" one gets:

$$\dot{y} = 1y - 2xy - 1yy + 2xyy$$

that coincides with the calculated result. In the Figure 15 we show a game, where the strategy 'non-aggressive' is dominated by the strategy 'aggressive'. In the phase plane one can see, how the aggressive population is the surviving species (converges to one)

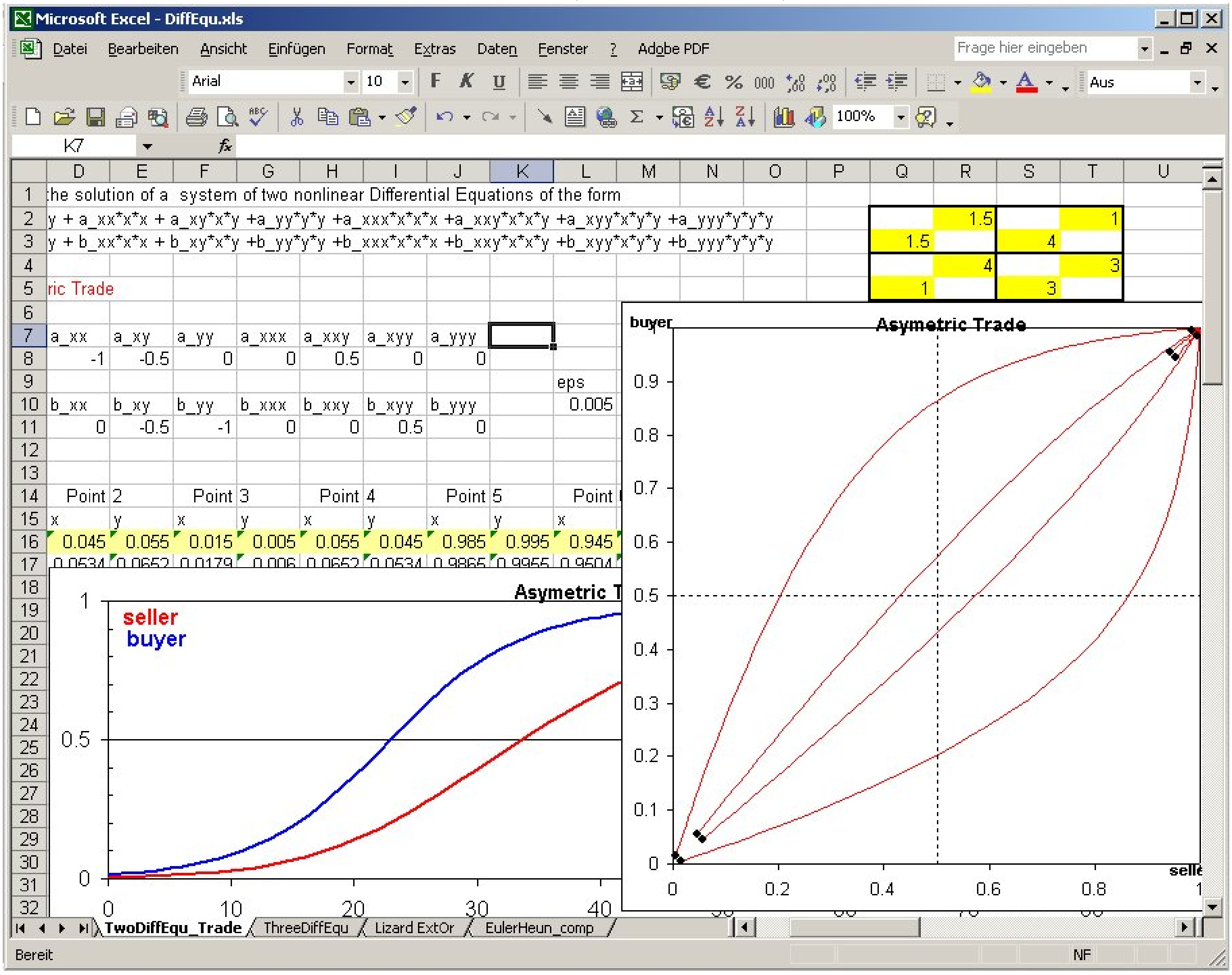


Figure 15: Asymmetric Trade, non-aggressive is dominated

3.3 Rock-paper-scissors game

We look at the children's game "Rock-paper-scissors" where each player chooses one of three strategies.
A rock beats a pair of scissors, a pair of scissors beats a sheet of paper and a sheet of paper beats a rock.

		Player 2		
		Paper	Scissors	Rock
Player 1	Paper	0 \ 0 (I)	-1 \ 1 (II)	1 \ -1 (III)
	Scissors	1 \ -1 (IV)	0 \ 0 (V)	-1 \ 1 (VI)
	Rock	-1 \ 1 (VII)	1 \ -1 (VIII)	0 \ 0 (IX)

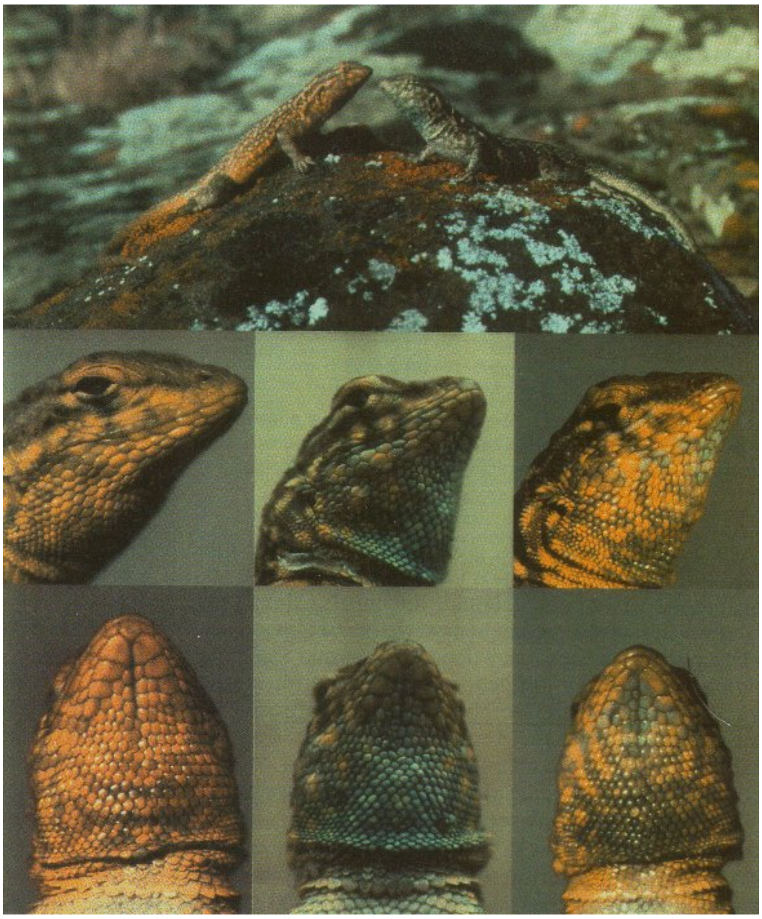
Figure 16: Rock-paper-scissors game

3.4 Side-blotched lizard

B. Sinervo and C. M. Lively examine in their publication "The rock-paper-scissors game and the evolution of alternative male strategies" [Sinervo und Lively 1996, p. 240-242] the mating behaviour of the side-blotched lizard and argue:

"We have described the first biological example of a cyclical 'Rock-paper-scissors' game. As in the game where paper beats rock, scissors beat paper, and rock beats scissors, the wide-ranging 'ultra-dominant' strategy of yellow males, which is in turn defeated by the mate-guarding strategy of blue males; the orange strategy defeats the blue strategy to complete the dynamic cycle. Frequency-dependent selection maintains substantial genetic variation in alternative male strategies, while at the same time prohibiting a stable equilibrium in morph frequency." ([Sinervo und Lively 1996, p. 241])

Using empirical findings Sinervo and Lively conclude:



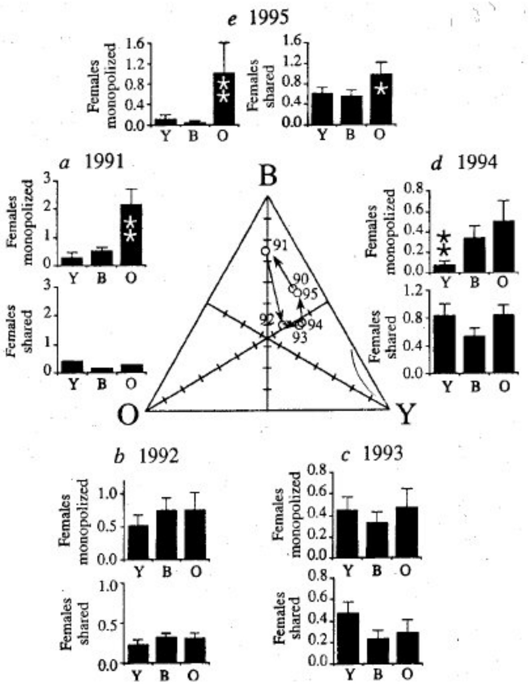
Color polymorhisms of male side-blotched lizards. Figure 1 from [Sinervo und Lively 1996, p. 241]

"We observed significant frequency-dependence in every year (see text). Partial regression coefficients (adjusted to relative fitness) describing gains and losses are given by:

$$N(G_M+G_s)=\begin{bmatrix}5.03 & 0 & 0\\ 0 & 2.35 & 0\\ 0 & 0 & 2.95\end{bmatrix}$$

$$\times\left(\begin{bmatrix}0 & 0 & +0.15\\ 0 & 0 & -0.07\\ 0 & 1.24 & 0\end{bmatrix}+\begin{bmatrix}0 & 0 & +0.35\\ +0.27 & 0 & 0\\ -0.93 & 0 & 0\end{bmatrix}\right)$$

For example, a rare yellow would gain 0.15 females by monopoly and 0.35 females by sharing when playing against common orange males. A rare yellow's gains in relative fitness are equal to $5.03 \times (0.15 + 0.35)$, given between rare and common neighbours."([Sinervo und Lively 1996, p. 241])



Among-years changes in the frequency of adult male colour polymorphisms.
Figure 2 from [Sinervo und Lively 1996, p. 241]

This results in

$$N(G_M+G_S) = \begin{bmatrix}5.03 & 0 & 0\\ 0 & 2.35 & 0\\ 0 & 0 & 2.35\end{bmatrix}\begin{bmatrix}0 & 0 & 0.5\\ 0.27 & 0 & -0.07\\ -0.93 & 1.24 & 0\end{bmatrix} = \begin{bmatrix}0 & 0 & 2.515\\ 0.6345 & 0 & -0.1645\\ -2.7435 & 3.658 & 0\end{bmatrix}$$

For this situation we want to calculate replicator equations.

3.5 Replicator equations

Specific values

First we will use the specific values of Sinervo-Lively and transform them to a game matrix
This game matrix, shown in Figure 17 corresponds to a certain degree to a "rock-paper-scissors"-game.

$$u(g) = x_g \cdot 0 + x_b \cdot 0 + x_o \cdot 2.515$$
$$u(b) = x_g \cdot 0.5 + x_b \cdot 0 + x_o \cdot (-0.1)$$
$$u(o) = x_g \cdot (-2.7) + x_b \cdot 3.6 + x_o \cdot 0$$

$$\dot{x}_g = x_g[u_g - \{x_g u_g + x_b u_b + x_o u_o\}]$$

	yellow (g)	blue (b)	orange (o)
yellow (g)	0 / 0	0.6345 / 0	-2.7435 / 2.515
blue (b)	0 / 0.6345	0 / 0	3.658 / -0.1645
orange (o)	2.515 / -2.7435	-0.1645 / 3.658	0 / 0

Figure 17: Side-blotched Lizard Game

$$\begin{array}{rclccccccc} \dot{x}_g & = & x_g[x_o \cdot 2.515 - \{ & x_g \cdot x_g \cdot 0 & + & x_g \cdot x_b \cdot 0 & + & x_g \cdot x_o \cdot 2.515 & + \\ & & & x_b \cdot x_g \cdot 0.6345 & + & x_b \cdot x_b \cdot 0 & + & x_b \cdot x_o \cdot (-0.1645) & + \\ & & & x_o \cdot x_g \cdot (-2.7435) & + & x_o \cdot x_b \cdot 3.658 & + & x_o \cdot x_o \cdot 0 & \}] \end{array}$$

As $x_o = 1 - x_g - x_b$ it follows:

$$\begin{array}{lccccc} \dot{x}_g = x_g[2.515(1-x_g-x_b) - \{ & 0 & + & 0 & + & 2.515x_g(1-x_g-x_b) \\ & +0.6345x_bx_g & + & 0 & - & 0.1645x_b(1-x_g-x_b) \\ & -2.7435(1-x_g-x_b)x_g & + & 3.658(1-x_g-x_b)x_b & + & 0 \quad \}] \end{array}$$

$$\begin{array}{ll} \dot{x}_g = x_g & [2.515 - 2.515x_g - 2.515x_b \\ & -2.515x_g + 2.515x_gx_g + 2.515x_gx_b \\ & -0.6345x_bx_g + 0.1645x_b - 0.1645x_bx_g - 0.1645x_bx_b \\ & +2.7435x_g - 2.7435x_gx_g - 2.7435x_bx_g - 3.658x_b + 3.658x_gx_b + 3.658x_bx_b] \end{array}$$

$$\begin{array}{ll} \dot{x}_g = x_g & [2.515 + (2.7435 - 2.515 - 2.515)x_g + (0.1645 - 2.515 - 3.658)x_b \\ & +(2.515 - 2.7435)x_gx_g + (2.515 - 0.6345 - 0.1645 - 2.7435 + 3.658)x_gx_b \\ & +(3.658 - 0.1645)x_bx_b \qquad] \end{array}$$

$$\dot{x}_g = \quad 2.515x_g - 2.2865x_gx_g - 6.0085x_gx_b \quad -0.2285x_gx_gx_g + 2.6305x_gx_gx_b \quad +3.4935x_gx_bx_b$$

The calculation of $\dot{x}_b$ starts from

$$\dot{x}_b = x_b[u_b - \{x_gu_g + x_bu_b + x_ou_o\}]$$

This only differs from the calculation of $\dot{x}_g$ by the term u_b. We therefore get at once

$$\begin{array}{ll} \dot{x}_b = x_b & [0.6345x_g - 0.1645 + 0.1645x_g + 0.1645x_b \\ & -2.515x_g + 2.515x_gx_g + 2.515x_gx_b \\ & -0.6345x_bx_g + 0.1645x_b - 0.1645x_bx_g - 0.1645x_bx_b \\ & +2.7435x_g - 2.7435x_gx_g - 2.7435x_bx_g - 3.658x_b + 3.658x_gx_b + 3.658x_bx_b] \end{array}$$

and from that

$$\dot{x}_b = x_b \quad [-0.1645 + 1.0275x_g - 3.329x_b \quad -0.2285x_gx_g + 2.6305x_gx_b \quad +3.4935x_bx_b]$$

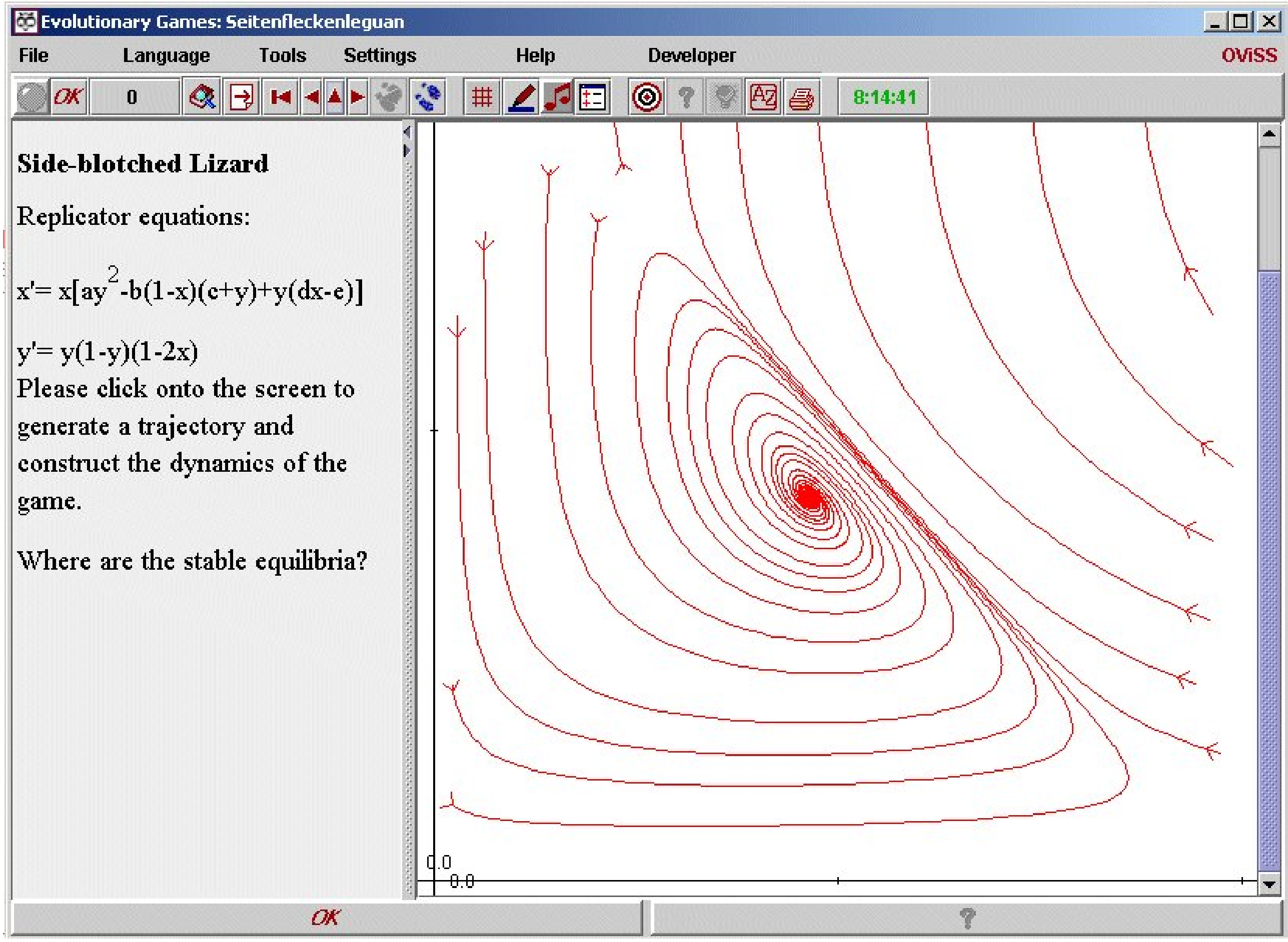


Figure 18: The Side-blotched lizard as an applet

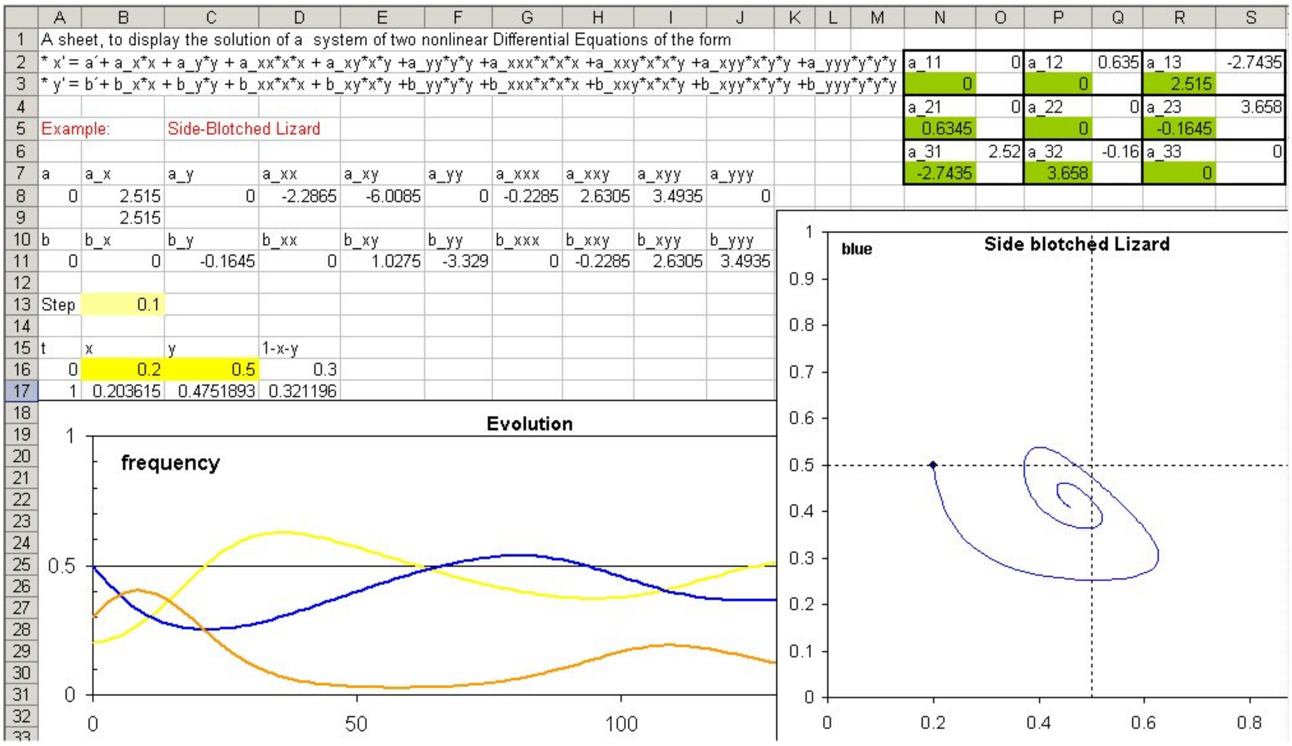


Figure 19: The Side-blotched lizard in a spreadsheet

General values

We now start from general values, given by the matrix and calculate the utility of a player of i, if there are a number x_j players of type j.

$$\begin{array}{lclclcl} u_1 \stackrel{def}{=} u(1) & = & a_{11}x_1 & + & a_{12}x_2 & + & a_{13}x_3 \\ u_2 \stackrel{def}{=} u(2) & = & a_{21}x_1 & + & a_{22}x_2 & + & a_{23}x_3 \\ u_3 \stackrel{def}{=} u(3) & = & a_{31}x_1 & + & a_{32}x_2 & + & a_{33}x_3 \end{array}$$

		Type 1	Type 2	Type 3
Type	1	a_{11}	a_{12}	a_{13}
	2	a_{21}	a_{22}	a_{23}
	3	a_{31}	a_{32}	a_{33}

Figure 20: Side-blotched Lizard Game

We get replicator equations by comparing the payoffs for a player of type i with the average payoff to all players.

We find:

$$\begin{array}{lcl} \dot{x}_i & = & x_i(u_i - \{x_1u_1 + x_2u_2 + x_3u_3\}) \\ & = & x_i[a_{i1}x_1 + a_{i2}x_2 + a_{i3}x_3 \\ & - & \{x_1a_{11}x_1 + x_1a_{12}x_2 + x_1a_{13}x_3 + x_2a_{21}x_1 + x_2x_{22}x_2 + x_2a_{23}x_3 \\ & + & x_3a_{31}x_1 + x_3a_{32}x_2 + x_3a_{33}x_3\}] \\ & = & [a_{i1}x_1 + a_{i2}x_2 + a_{i3}x_3 \\ & - & \{a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 + (a_{12}+a_{21})x_1x_2 + (a_{13}+a_{31})x_1x_3 + (a_{23}+a_{32})x_2x_3\}] \end{array}$$

As the three shares add to 1, we get:

$$\begin{array}{lclcl} \dot{x}_i & = & x_i[a_{i1}x_1 + a_{i2}x_2 & + & a_{i3}(1 - x_1 - x_2) \\ & & -(a_{12}+a_{21})x_1x_2 & - & (a_{13}+a_{31})x_1(1 - x_1 - x_2) \\ & & & - & (a_{23}+a_{32})x_2(1 - x_1 - x_2)] \\ & = & x_i[(a_{i1} - a_{i3})x_1 & + & (a_{i2} - a_{i3})x_2 + a_{i3} \\ & - & (a_{12}+a_{21})x_1x_2 & - & (a_{13}+a_{31})x_1 + (a_{13}+a_{31})x_1x_1 + (a_{13}+a_{31})x_1x_2 \\ & - & (a_{23}+a_{32})x_2 & + & (a_{23}+a_{32})x_1x_2 + (a_{23}+a_{32})x_2^2] \end{array}$$

$$\begin{array}{lllcl} = & x_i[a_{i3} & + & (a_{i1} - a_{i3} - a_{13} - a_{31})x_1 \\ & & + & (a_{i2} - a_{i3} - a_{23} - a_{32})x_2 \\ & & + & (a_{13}+a_{31})x_1^2 \\ & & + & (a_{23}+a_{32})x_2^2 \\ & & + & (-a_{12} - a_{21} + a_{13} + a_{31} + a_{23} + a_{32})x_1x_2] \end{array}$$

These are the replicator equations for populations 1 and 2, in which we are interested.

For a check, we insert the empirical values of Sinervo/Lively and get:

$$\dot{x}_1 = 2.515x_1 - 2.2865x_1^2 - 6.0085x_1x_2 - 0.2285x_1^3 + 2.6305x_1^2x_2 + 3.4935x_1x_2^2$$

und

$$\dot{x}_2 = 0.1645x_2 + 1.0275x_1x_2 - 3.329x_2^2 - 0.2285x_1^2x_2 + 2.6305x_1x_2^2 + 3.4935x_2^2$$

This confirms the formulas which we derived in this section.

The differential equations generate cyclic-convergent orbits and thus the empirical values observed by Sinervo/Lively.

However, variations of the values inside the game matrix will generate different evolutions and with that also for example stable cycles or diverging cycles. With diverging cycles, however, the extinction of some of the species would emerge.

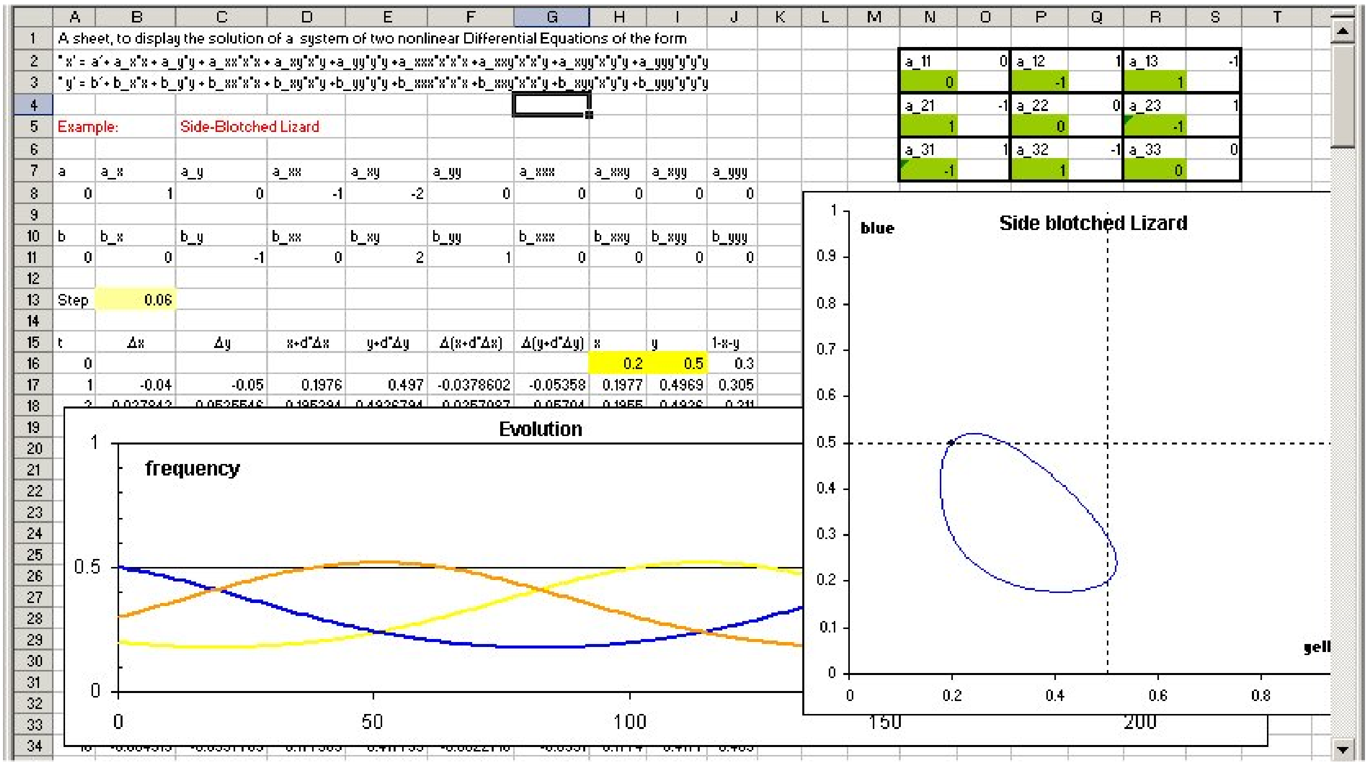


Figure 21: Side-blotched Lizard, Cycle

3.6 Heun's Method – The Improved Euler's Method

Euler's method has its problems. We saw that already in the demonstrations: In many cases it doesn't work sufficiently exact. Let us have again a very close look at the method at it's work. For this we have chosen an example, where we know the correct trajectories and where those are constantly doing a left turn with a (nearly) constant distance. We have chosen a rather large step for demonstration. Then we can see clearly the problem of the method. At point x_0 we calculate a new point x_1 just by constructing the tangent at x_0 and going the step-length along that tangent. As the trajectories are bended, the new point will always lie on another trajectory. From x_1 we calculate in the same manner the next point, which again lies away from the trajectory of x_1 and even further away from the original trajectory.
We can improve on this by choosing a shorter step but the error will remain.

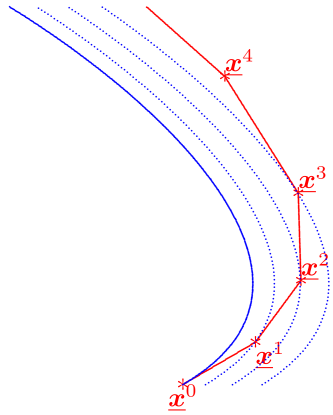


Figure 22: Euler's method

The improved Euler's method does a far better job. It works like this. As with Euler's method one calculates from $\underline{x}^0$ beginning a new point $\underline{x}^{01}$ and from there another one $\underline{x}^{02}$. Those points however are not new predictions for the trajectory but tentative ones only used to get a corrected prediction. For this we connect the points $\underline{x}^0$ and $\underline{x}^{02}$ by a straight line and define the midpoint $\underline{x}^0 = 0.5\underline{x}^0 + 0.5\underline{x}^{02}$ as the next predictor for the trajectory. As one can see in the figure, this 'predictor-corrector algorithm' gives a far better fit than the 'predictor algorithm' of Euler's method. Even with a really huge step-length, the predicted point are rather near to the true trajectory.

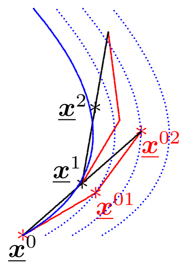


Figure 23: Heun's method

Now we will implement Heun's method in a spreadsheet. Of course, it is more tiring than Euler's method but not really complicated:
We start with the same Excel sheet, which we have used above with the named entries in row 8 and 11. We change only the rows 15 to 17.

	A	B	C	D	E	F	G	H	I	J
7	a	a_x	a_y	a_xx	a_xy	a_yy	a_xxx	a_xxy	a_xyy	a_yyy
8	0	3	-5	0	0	0	0	0	0	0
9										
10	b	b_x	b_y	b_xx	b_xy	b_yy	b_xxx	b_xxy	b_xyy	b_yyy
11	0	5	-3	0	0	0	0	0	0	-1
12										
13	Step d=	0.1								
14										
15	t	Δx	Δy	$x+d*\Delta x$	$y+d*\Delta y$	$\Delta(x+d*\Delta x)$	$\Delta(y+d*\Delta y)$	x	y	
16	0						0.2	0.2		
17		(⊛)	(⊛)	(⊛)	(⊛)	(⊛)	(⊛)	(⊛)	(⊛)	

Now we enter the following formulas into row 17:

In B17 we calculate the x-differences from the given point $\underline{x}^t$ to the first (tentative) predictor $\hat{\underline{x}}^{t+1}$:
=(a+a_x*H16+a_y*I16+a_xx*H16^2+a_xy*H16*I16+a_yy*I16^2+a_xxx*H16^3+a_xyy*H16*I16^2+a_xxy*H16^2*I16+a_yyy*I16^3)

In C17we calculate the y-differences from the given point $\underline{\boldsymbol{x}}^t$ to the first (tentative) predictor $\hat{\underline{\boldsymbol{x}}}^{t+1}$:

=(b+b_x*H16+b_y*I16+b_xx*H16^2+b_xy*H16*I16+b_yy*I16^2+b_xxx*H16^3+b_xyy*H16*I16^2+b_xxy*H16^2*I16+b_yyy*I16^3)

In D17 we calculate x-coordinate of the first (tentative) predictor $\hat{\underline{\boldsymbol{x}}}^t$:

=H16+d*B17

In E17 we calculate y-coordinate of the first (tentative) predictor $\hat{\underline{\boldsymbol{x}}}^t$:

=I16+d*C17

In F17 we calculate the x-differences from the first (tentative) predictor $\hat{\underline{\boldsymbol{x}}}^{t+1}$ to the second (tentative) predictor $\hat{\underline{\boldsymbol{x}}}^{t+2}$:

=(a+a_x*D17+a_y*E17+a_xx*D17^2+a_xy*D17*E17+a_yy*E17^2+a_xxx*D17^3+a_xyy*D17*E17^2+a_xxy*D17^2*E17+a_yyy*E17^3)

In G17 we calculate the y-differences from the first (tentative) predictor $\hat{\underline{\boldsymbol{x}}}^{t+1}$ to the second (tentative) predictor $\hat{\underline{\boldsymbol{x}}}^{t+2}$:

=(b+b_x*D17+b_y*E17+b_xx*D17^2+b_xy*D17*E17+b_yy*E17^2+b_xxx*D17^3+b_xxy*D17^2*E17+b_xyy*D17*E17^2+b_yyy*E17^3)

In H17 we calculate the x-coordinate of the next (real) predictor $\underline{\boldsymbol{x}}^{t+1}$
=H16+d*(B17+F17)/2

In I17 we calculate the x-coordinate of the next (real) predictor $\underline{\boldsymbol{x}}^{t+1}$
=I16+d*(C17+G17)/2

4 More on Dynamics

In the following, we will give a series of other examples on dynamics in the program. Evidently, it is impossible to show all the steps and methods employed. Instead, emphasis is placed on the particulars of the structure and especially the usefulness of these concepts for other fields of education.

4.1 Tournaments

Another fascinating area of Game Theory is the "Evolution of Cooperation" [Axelrod 1984]:(Axelrod 1984). The next two figures show examples from OViSS, which deal with this field. The first shows a repeated Dilemma situation between two players, playing different strategies. The second shows the graphical representation of the outcome of a complete tournament.

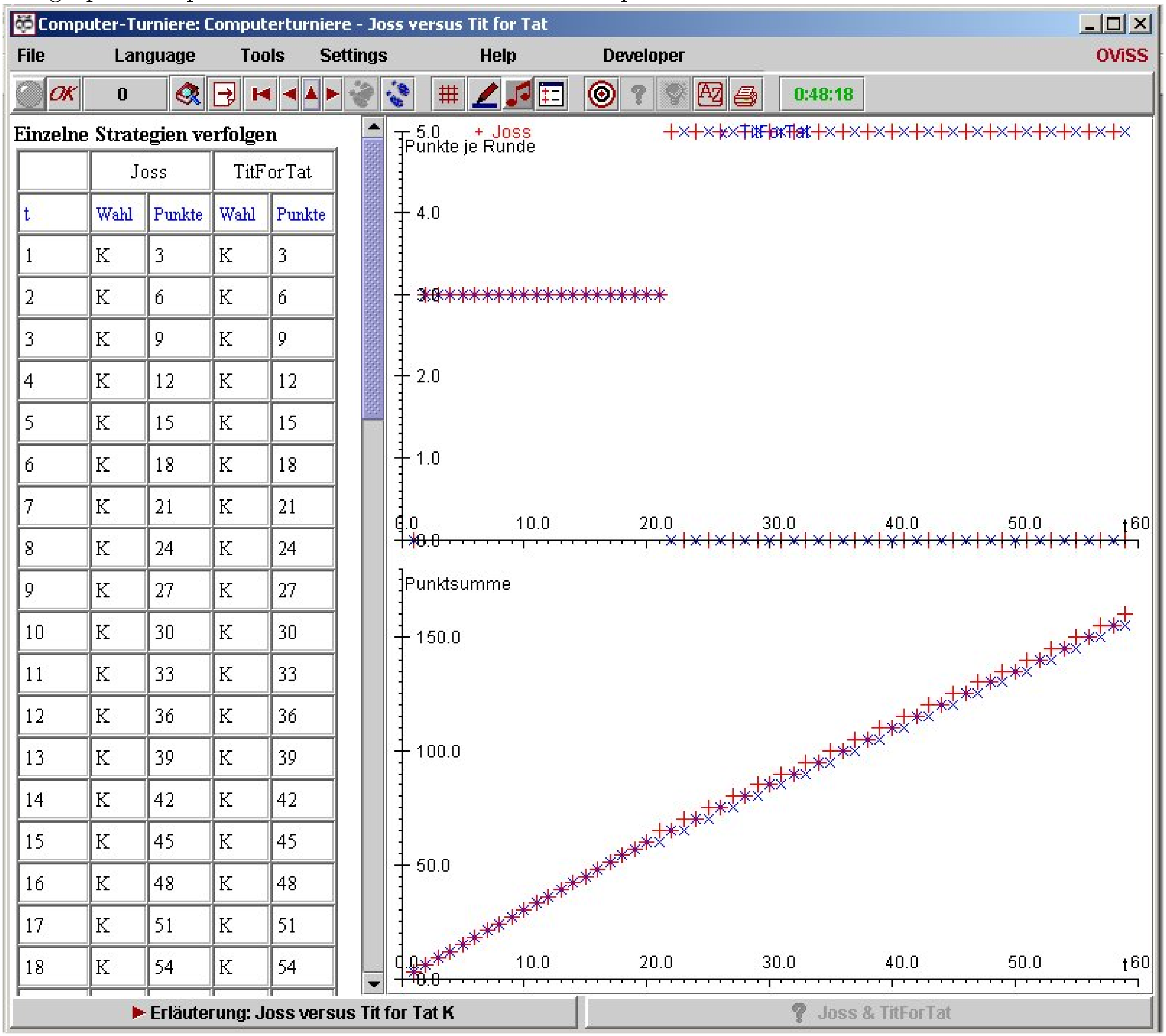


Figure 24: Joss versus TitForTatK

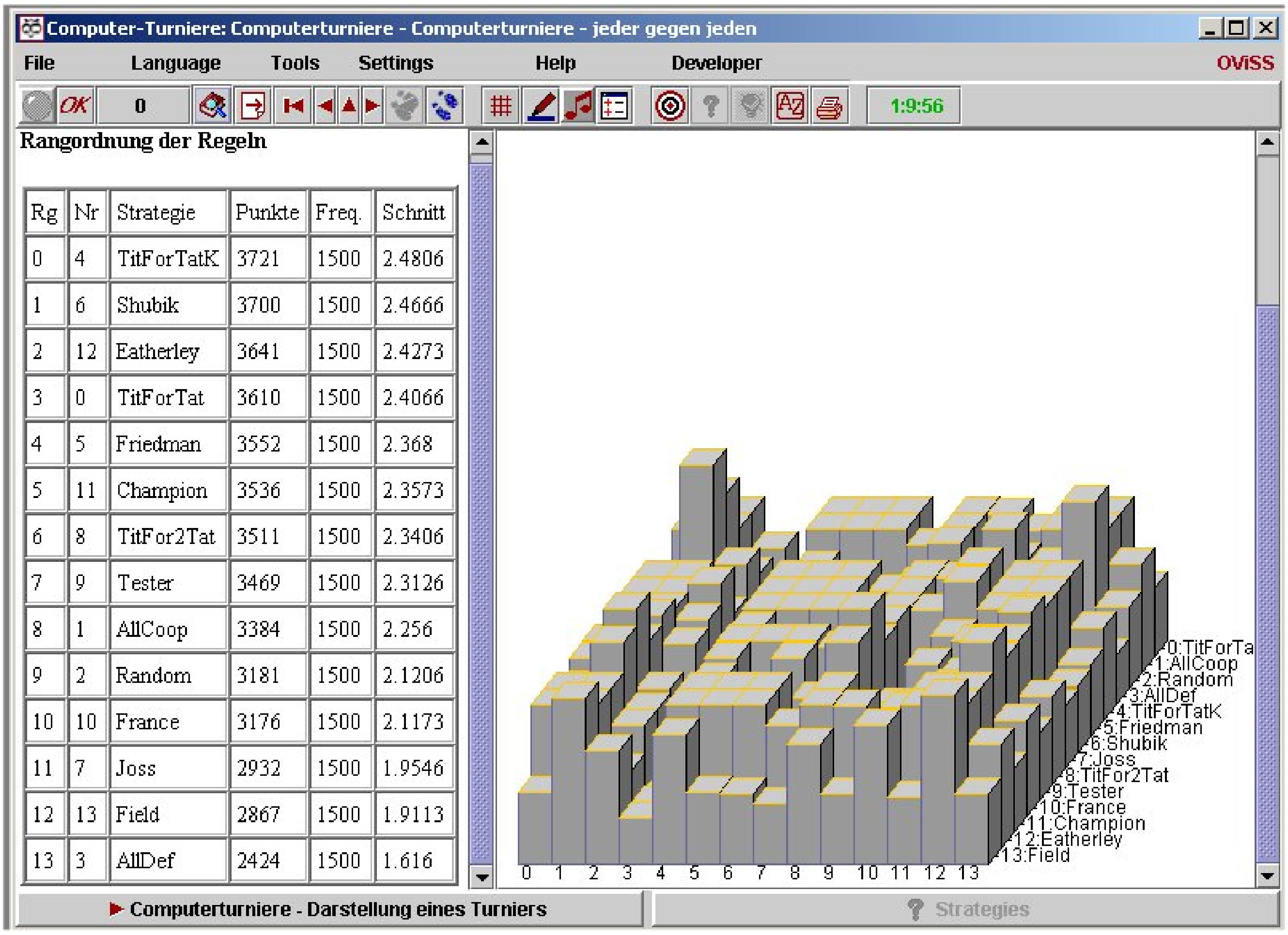


Figure 25: Each versus Everyone

4.2 Examples from Market Dynamics

Kaldor Business Cycles

In 1940 Kaldor published his renowned model of the trade cycle [Kaldor 1940]. The purpose of this model was to show, by means of a simple diagrammatic apparatus, the necessary and sufficient conditions under which the combined operations of the multiplier and the investment demand function give rise to a business cycle. In the Chaos-Book of the OViSS system, Kaldor's business cycle model and some extensions are illustrated.

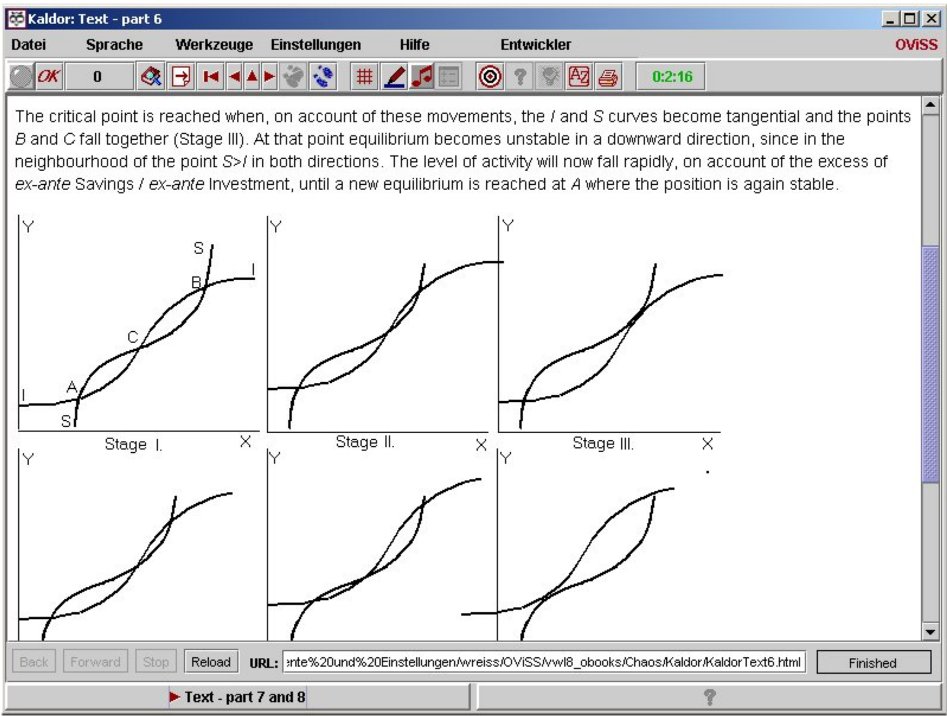


Figure 26: Part of the Original Kaldor Text

The user should especially concentrate on the following tasks:

1. He/she should read the original text and try to understand the dynamics.
2. He/she should watch the adjustments of the investment function and the saving function of the original Kaldor model as a dynamic process.
3. He/she should watch the emerging business cycle in the phase space as well as in the time domain (time series of production and investment).
4. He/she should reexamine the Kaldor model in the way proposed by [Dana und Malgrange 1984] and [Lorenz 1984]and notice that for certain parameters the Kaldor model can give rise to chaotic behaviour. This chaotic behaviour is again demonstrated by displaying the underlying dynamic process and the trajectories in the phase space and the time series.

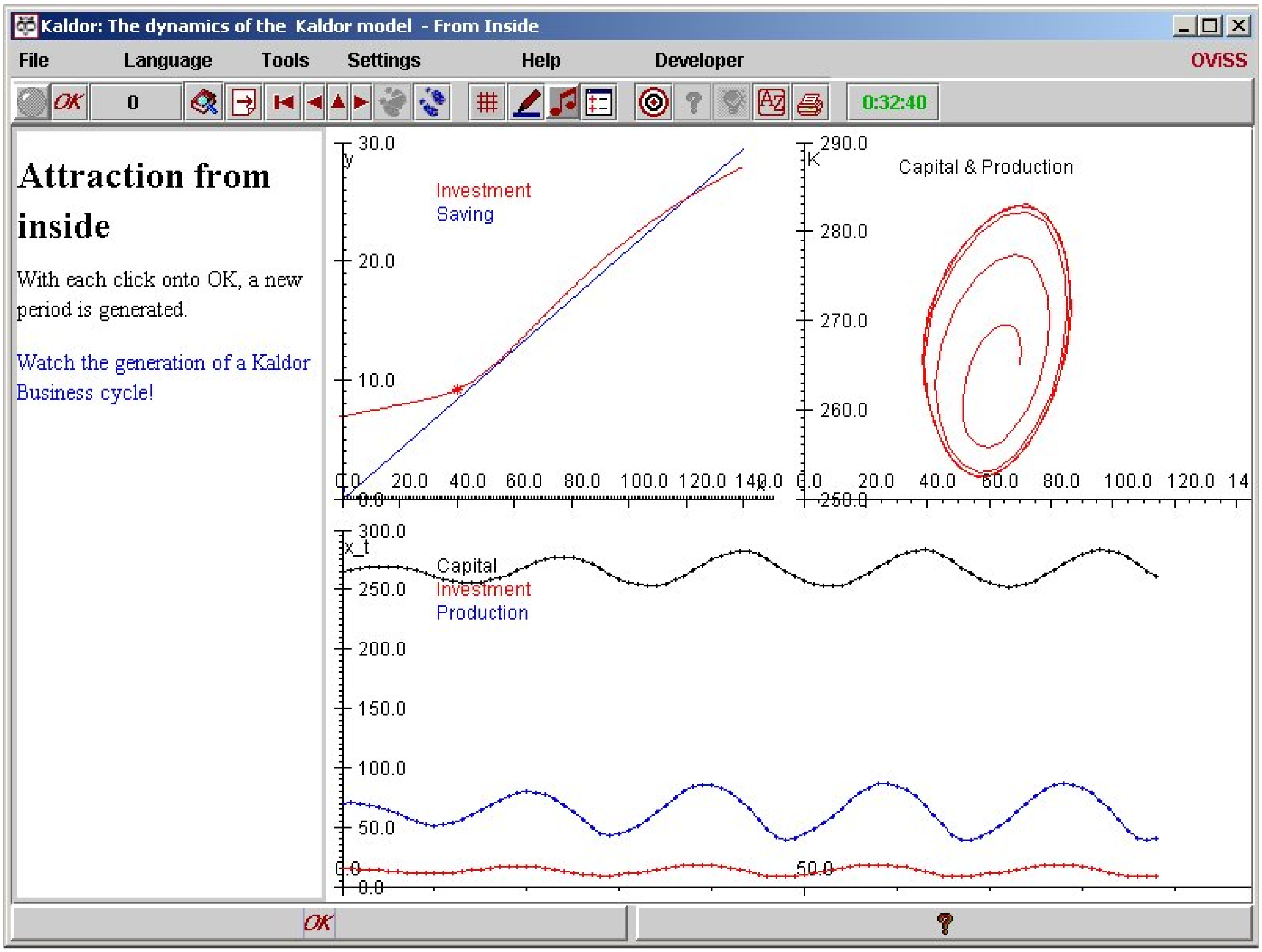


Figure 27: The Kaldor Business Cycle

4.3 The Tâtonnement Process in an Exchange Economy

This unit shows a tâtonnement process. The users can watch the calculations and the adjustments in an Edgeworth box. The underlying model of the process is explained and the structure is shown in the help topics.

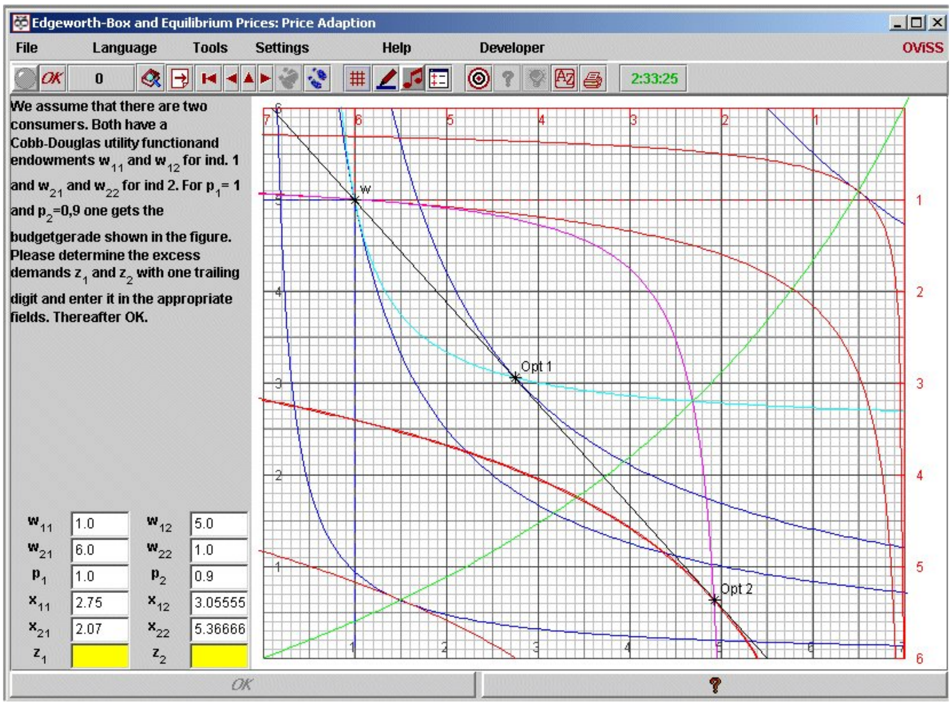


Figure 28: Tâtonnement in an Exchange Economy – start

The users can watch the tâtonnement process in the Edgeworth box as well as in a table which displays the formal model.

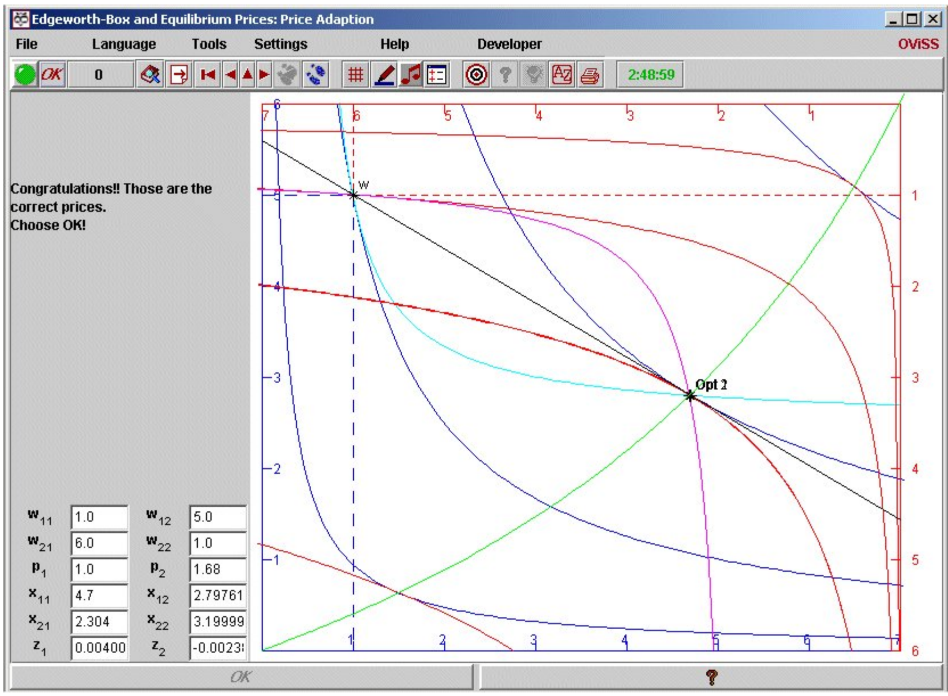


Figure 29: Tâtonnement in an Exchange Economy – equilibrium

5 Summary

The purpose of this paper was to show the possible use of OViSS for teaching interactively dynamics in economics. For this we demonstrated, how the orbits of a system of nonlinear differential equations can be presented inside a spreadsheet. To guarantee an easy access for the students, we first used Euler's method for the construction of the orbits but improved that procedure later on by Heun's method.
The usefulness of this concept was demonstrated for different examples from economics. These concepts, however, can easily be applied to other areas of economics, as well as to other fields of science.

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